Examples

1. Auctions. For example $x = (p, t)$ where $p = (p_1, \ldots, p_I)$ is probability of winning and $t = (t_1, \ldots, t_I)$ is a vector of payments.
   
   $u_i(x, \theta_i) = \theta_i p_i - t_i$ (risk neutral, private valuation).
   
   The social choice function could be to give the item to the agent with the highest valuation at a price equal to the second highest valuation. (It could also be to give the item to the first agent and charge the second agent 100.)

2. Public Good.
   
   $x = (k, t)$ where $k$ is a binary variable (0 when you don’t build the bridge; 1 when you do). $t_i$ is $i$’s payment. $X$ constrained so that transfers sufficient to pay for the project.
   
   $u_i(x, \theta_i) = \theta_i k + t_i$

3. Adverse Selection
4. Monopoly Screening
5. Regulation Questions
6. Bargaining
7. Design of “Institutions”
Definition

A mechanism is \( \Gamma = (S_1, \ldots, S_i, g) \), where \( g : S \rightarrow X \).

Given \( \Gamma \) and preferences you obtain a game:

- \( I \) players.
- \( S \) strategy space.
- \( U_i(s, \theta_i) = u_i(g(s), \theta_i) \) payoffs.
Given a social choice problem, can you “implement” it while preserving incentives. $\Gamma$ implements $f$ if there is an equilibrium $s^*$ of the game induced by $\Gamma$ such that $g(s^*(\theta)) = f(\theta)$. Variations: Change solution concept (“dominant strategy implementation”) or require that induced game has unique eq. Set valued $f$. Alternative: Maximize objective subject to incentive constraints.
Huh?

This formulation
Two Directions

One literature starts with a social welfare function and asks: Can this be implemented?

1. In GE environment, is it possible to come up with a way to generate the Walrasian Correspondence?
2. In auctions, is it possible to always give item to highest valuation buyer?
3. A variety of (incomplete information) variations on classical social choice problems.

Another literature treats Mechanism Design as a way to study generalize agency problems. Mechanism Designer is Principal with the objective of maximizing something (profit, social welfare, etc). I’ll give an example of this.
Direct Mechanisms

Observe that design problem is hard (the space of mechanisms is huge). Critical simplifying assumption:
Without loss of generality, one can concentrate on direct and truthful mechanisms (direct: $S_i = \Theta_i$ for all $i$; $g(\theta) = f(\theta)$; truthful: an equilibrium in which $s_i^*(\theta_i) \equiv \theta_i$).

1. This is useful: Set of mechanisms is not manageable.
2. This is sensible: How could it be useful to ask agents to “report” more than their type (what else do they know?)?
3. This is amazing: Why would agents truthfully reveal their type?
Revelation Principal

If there exists a mechanism $\Gamma = (S_1, \ldots, S_I, g(\cdot))$ that implements the social choice function $f(\cdot)$ in Bayesian Nash equilibrium, then $f(\cdot)$ is truthfully implementable in BNE.

- By assumption we have $s^*$, BNE such that $g(s^*(\theta)) = f(\theta)$.
- Define new mechanism $\tilde{\Gamma} = (\Theta_1, \ldots, \Theta_I, f \circ s^*(\cdot))$.
- $s^*$ BNE for original mechanism implies truth BNE for $\tilde{\Gamma}$:

$$E_{\theta - i}[u_i(g(s_i^*(\theta_i), s_{-i}^*(\theta_{-i})), \theta_i) \mid \theta_i] \geq E_{\theta - i}[u_i(g(s_i^*(\hat{\theta}_i), s_{-i}^*(\theta_{-i})), \theta_i) \mid \theta_i]$$

follows from 1.

So the revelation principal is trivial. Central Insight:
You can assume that agents are honest if the mechanism designer / principal can commit to ignoring revealing information. In practice, the principal promises to “lie for” agents in order to get them to tell the truth. There are also equilibrium constraints that say that agents prefer to honestly report their type than to pretend to be another type.
Auction Example

- $I$ bidders.
- Action: $(y, t)$ where $y = (y_1, \ldots, y_I)$ describes the distribution probabilities ($y_i$ is the probability that $i$ wins); $t = (t_1, \ldots, t_I)$ is transfer vector.

\[ \bar{y}_i(\theta'_i) = E_{\theta_{-i}}[y_i(\theta'_i), \theta_{-i}] \text{ and } \bar{t}_i(\theta'_i) = E_{\theta_{-i}}[t_i(\theta_i), \theta_{-i}] \]

- $u_i(y) = \theta_i y_i + t_i$.
- $U_i(\text{theta}_i; \theta_i) = \theta_i \bar{y}_i(\theta'_i) + \bar{t}_i(\theta'_i)$ is the utility of a $\theta_i$ type who pretends to be $\theta'_i$ and other types are honest.
- $U_i(\theta_i) = \theta_i \bar{y}_i(\theta_i) + \bar{t}_i(\theta_i)$ is the utility of a $\theta_i$ type under truthful revelation.

- $-Et(\theta)$ is seller’s profit (under truthful revelation).


**Simplifying Constraints**

In order for truth-telling to be an equilibrium you need: \( U_i(\theta'_i, \theta) \) maximized at \( \theta'_i = \theta \) for all \( \theta \).

Algebraically:

\[
U_i(\theta) = \theta_i \bar{y}_i(\theta_i) + \bar{t}_i(\theta_i) \geq \theta_i \bar{y}_i(\theta_i) + \bar{t}_i(\theta_i) = U_i(\theta') + (\theta_i - \theta'_i)\bar{y}_i(\theta_i)
\]

or

\[
U_i(\theta) - U_i(\theta') \geq (\theta_i - \theta'_i)\bar{y}_i(\theta_i).
\]

By symmetry:

\[
U_i(\theta') - U_i(\theta) \geq (\theta'_i - \theta_i)\bar{y}_i(\theta_i').
\]

Hence:

\[
\bar{y}_i(\theta'_i) \geq \frac{U_i(\theta'_i) - U_i(\theta_i)}{\theta'_i - \theta} \geq \bar{y}_i(\theta_i)
\]

when \( \theta'_i > \theta_i \).
Implications

The incentive compatibility condition implies

$$\bar{y}_i(\theta_i') \geq \frac{U_i(\theta_i') - U_i(\theta_i)}{\theta_i' - \theta} \geq \bar{y}_i(\theta_i)$$

when $\theta_i' > \theta_i$, which in turn implies that

- $\bar{y}_i(\cdot)$ is nondecreasing.
- $U_i(\theta_i) = U_i(\theta_i) + \int_{\theta_i}^{\theta_i'} \bar{y}_i(s)ds$. 
\[
\begin{align*}
\text{• } \bar{y}_i(\cdot) & \text{ is nondecreasing.} \\
\text{• } U_i(\theta_i) = U_i(\bar{\theta}_i) + \int_{\theta_i}^{\theta_i} \bar{y}_i(s) ds.
\end{align*}
\]

imply incentive compatibility. For \( \theta_i' > \theta_i \)

\[
U_i(\theta_i') - U_i(\theta_i) = \int_{\theta_i}^{\theta_i'} \bar{y}_i(s) ds \geq \int_{\theta_i}^{\theta_i'} \bar{y}_i(\theta_i) ds = (\theta_i' - \theta)\bar{y}_i(\theta_i)
\]
Simplified Auction Problem

Objective function: \( \sum_{i=1}^{l} E[-t_i(\theta)] \). Look at

\[
E[-t_i(\theta)] = E_{\theta_i}[-\bar{t}_i(\theta_i)] = \int_{\bar{\theta}_i}^{\theta_i} \left[ \tilde{y}_i(\theta_i)\theta_i - U_i(\theta_i) \right] \phi_i(\theta_i) d\theta_i
\]

\[
\int_{\bar{\theta}_i}^{\theta_i} \left( \tilde{y}_i(\theta_i)\theta_i - U_i(\theta_i) - \int_{\theta_i}^{\theta_i} (\tilde{y}_i(s)ds) \right) \phi_i(\theta_i) d\theta_i =
\]

\[
\left[ \int_{\bar{\theta}_i}^{\theta_i} \left( \tilde{y}_i(\theta_i)\theta_i - \int_{\theta_i}^{\theta_i} (\tilde{y}_i(s)ds) \right) \phi_i(\theta_i) d\theta_i \right] - U_i(\theta_i).
\]

You can write:

\[
\int_{\bar{\theta}_i}^{\theta_i} \left( \int_{\theta_i}^{\theta_i} \tilde{y}_i(s)ds \right) \phi_i(\theta_i) d\theta_i = \int_{\bar{\theta}_i}^{\theta_i} \tilde{y}_i(\theta_i) d\theta_i - \int_{\theta_i}^{\theta_i} \tilde{y}_i(\theta_i) \Phi_i(\theta_i) d\theta_i = \int_{\bar{\theta}_i}^{\theta_i} \tilde{y}_i(\theta_i) d\theta_i
\]

and so \( E[-\bar{t}_i(\theta_i)] =:\)

\[
-U_i(\theta_i) + \left[ \int_{\bar{\theta}_i}^{\theta_i} \tilde{y}_i(\theta_i) \left( \theta_i - \frac{1 - \Phi_i(\theta_i)}{\phi_i(\theta_i)} \right) \phi_i(\theta_i) d\theta_i \right]
\]
Revenue Equivalence Theorem

All of that algebra has an amazing consequence. We found that the revenue from any incentive compatible auction mechanism is a function of two things: the utility of the lowest type \( U_i(\theta_i) \) and the probability of winning function \( y \). Hence all auctions that agree on these quantities (and are incentive compatible) raise the same revenue.
In particular, suppose the auction always awards the item to the highest bidder and leaves the lowest types with zero utility, then they raise the same revenue. There are many different auctions that do this (for example first and second price auctions).
Optimal Auction

We know the objective function, the constraint is that $\bar{y}_i$ is monotonic. So to solve the problem we set $U_i(\theta_i) = 0$ (individual rationality) and try to maximize:

$$E\left[\sum_{i=1}^{l} y_i(\theta) \left( \theta_i - H_i(\theta_i) \right)\right]$$

, where

$$H_i(\theta_i) = \frac{1 - \Phi_i(\theta_i)}{\phi_i(\theta_i)}$$

is the hazard rate.

Given that the $y_i$ are non-negative and sum to one the solution to the problem without the monotonicity constraint would be to set $y_j(\theta) = 1$ when

$$\theta_j - H_j(\theta_j) = \max\{\theta_i - H_i(\theta_i)\}.$$  

This is the actual solution in interesting cases (symmetry and $\theta_i - H_i(\theta_i)$ monotonically increasing).