Economics 200C, Spring 2012
Practice Midterm Answer Notes

I tried to write questions that are in the same form, the same length, and the same difficulty as the actual exam questions. I failed. I think that the real exam will have more questions and the questions will need shorter, simpler answers. Rough guidelines: You are responsible for "everything," but I really want to make sure that you know the main ideas. Some of the main ideas are working with and manipulating definitions. Do you know what a strategy is? Can you go from strategy to outcome? Can you identify a best response? Can you identify an equilibrium? Do you know the difference between Nash and subgame perfect? What is the single-crossing condition? What is the difference between signaling and screening? Some of the main ideas involve understanding basic results. In my opinion, the main results are: the folk theorem of repeated games, the possibility of market failure with adverse selection, the structure of equilibria in signaling games (pooling versus separating), the efficiency properties of signaling equilibria, the non-existence of equilibrium in screening models. The questions below do not cover all of these topics, but combined with problems from the text and homework problems, I hope that you will be prepared for Wednesday's test.

1. The questions that follow refer to the repeated Prisoner's Dilemma game with stage-game payoff matrix:

	Cooperate	Defect
Cooperate	$5, 5$	$0, 8$
Defect	$8, 0$	$1, 1$

Consider the infinitely repeated Prisoner's Dilemma (players discount payoffs with the common discount factor $\delta = .75$).

(a) Does there exist a subgame-perfect equilibrium of the game in which both players cooperate in every period on the equilibrium path?

Consider the grim trigger strategies (start by cooperating, continue to cooperate if and only if history is completely cooperative). These strategies best-respond following any history containing a defection. Otherwise a player has a choice of following the equilibrium in order to receive the average payoff of 5 or deviating, which leads to an average payoff of $(1-\delta)8+\delta$. So these strategies are an equilibrium provided that $(1-\delta)8+\delta < 5$ or $\delta > \frac{3}{7}$. So the answer to the question is yes.

(b) Consider the strategy: Begin by cooperating. Cooperate after any history in which there has been no more than one period in which a player defected. In all other histories, defect. Call this strategy "Forgive Once."

i. Find a best response to the strategy Forgive Once. Is your strategy a best response to Forgive Once in every subgame? If so, find a best response that is not a best response to Forgive Once in every subgame. If not, find a best response that is a best response to Forgive Once in every subgame.

One best response is to cheat in the initial period and then cooperate in all future periods. The previous part demonstrates that you do not want to cheat twice (and trigger a punishment). The first defection is not punished and gains in the short term. Because you discount, you want to cheat once as early as possible. This strategy is a not a best response in all subgames. In particular, in a subgame in which there have been no defections, it is better to defect in the first period and cooperate in all subsequent periods (provided that there have been no other defections) and to defect if there has been more than one defection in the history.

ii. What is the outcome of the game if player plays Forgive Once and player two plays the best response identified above. Do the strategies Forgive Once and the best response from (i) constitute a Nash Equilibrium for the repeated Prisoner's Dilemma? If so, prove it. If not, explain why not.
The outcome is (D, C) in the first period and (C, C) is all future periods. It is not a Nash equilibrium (opponent would want to cheat if you never punish). On the other hand, other best responses (like cheat in the first period, cooperate thereafter if and only if opponent has always cooperated) do lead to Nash equilibria.

iii. Is the best response unique? If so, prove it. If not, exhibit another best response.
Not unique. Examples in previous answers.

(c) Suppose agents play the repeated prisoner's dilemma, but they do not observe the outcome of the stage game perfectly. Specifically, assume that if the stage game outcome is (s_1, s_2), then players observe (s_1, s_2) with probability γ and each of the other three outcomes with equal probability $(1-\gamma)/3$. Suppose both players play Forgive Once (Begin by cooperating. Cooperate after any history in which there has been no more than one period in which (C, C) has been observed). What is the expected payoff if both players play Forgive Once?

If both players play forgive once, then they will obtain identical payoffs. Let V denote these payoffs. Let V_1 denote the payoffs following a history in which there has been one observed deviation from (C, C). Let V_2 denote the payoff following a history in which there has been more than one observed deviation. For histories in which there has been more than one observed deviation, the outcome in all subsequent stages is (D, D) (I am assuming that even though there is noisy observations, the payoffs will be determined by actions selected. If payoffs themselves are noisy, then the expressions will be slightly more complicated, but follow the same basic form.) Hence $V_2 = 1$. $V_1 = (1-\delta)5 + \delta(\gamma V_1 + (1-\gamma)V_2)$. That is, in the first period, the player receives 5. In the future, the player gets either V_1 if there is no observed deviation from (C, C) (probability γ) and V_2 otherwise. Hence

$$V_1 = \frac{(1-\delta)5 + \delta(1-\gamma)}{1-\delta\gamma}.$$

Finally,

$$V = (1-\delta)5 + \delta(\gamma V + (1-\gamma)V_1).$$

It follows that

$$V = \frac{(1-\delta)5 + \delta(1-\gamma)V_1}{1-\delta\gamma}.$$

You are welcome to substitute the expression for V_1 and do further algebra.

2. A tutor offers to help a student prepare for an exam. If the student does not hire the tutor, then she will fail the class. If she hires the tutor, then the probability she passes the class depends on the quality of the tutor. If she hires a good tutor, she will pass with probability $\frac{3}{4}$. If she hires a bad tutor, she will pass with probability $\frac{1}{4}$. The student believes that the tutor is equally likely to be good or bad. Assume that the student receives the equivalent of a monetary prize of V if she passes the exam and 0 if she fails. She seeks to maximizes her expected payoff net of any fees. So, if she pays x to the

tutor and expects to pass the class with probability p, then her expected payoff is $pV - x$. The tutor seeks to maximize his expected earnings from tutoring. You may assume that both players use pure strategies.

(a) Find the (perfect Bayesian) equilibria of the game in which the tutor sets a price, which the student can either accept or reject. (If the student accepts the price x, then the tutor earns x and the student earns $pV - x$, where p is the probability of passing. If the student rejects, then both student and tutor earn zero.)

The problem says that you can look for pure strategy equilibria. So there is a pooling price or separating prices. Take pooling first. The tutor can earn at least $\frac{V}{4}$ because the student must accept any price strictly less than this (at the worst she passes with probability $\frac{1}{4}$). Hence any pooling price must be accepted with probability one. On the other hand, the expected value of a "pooled" tutor is $\frac{V}{2}$, so higher prices will be rejected. This means that the candidates for pooling equilibrium prices are any $x^* \in [\frac{V}{4}, \frac{V}{2}]$. There is an equilibrium outcome for any such x^*. A specification of an equilibrium that supports any of these as an equilibrium outcome is: Student accepts all prices less than x^* and rejects all higher ones. (Appropriate beliefs: the probability that the tutor is good given lower prices is $\frac{1}{2}$; the probability that the tutor is good given higher prices is 0.)

If the tutor's price depends on his type, the argument above implies that (in a pure strategy equilibrium) all offers must be accepted. But then no tutor would make the lower of the two offers. Hence no separation is possible. (There is a separating equilibrium outcome in which both types of tutor ask pV ($p = \frac{1}{4}$ or $\frac{3}{4}$) and the higher price is rejected with probability $\frac{2}{3}$.)

(b) Find the (perfect Bayesian) equilibria of the game in which the tutor sets a price and also may offer a "double your money back" rebate in case of failure, which the student can either accept or reject. (If the student accepts the price x with a guarantee, then the tutor earns $x - 2(1-p)x$ and the student earns $p(V - x) + (1 - p)x$, where p is the probability of passing. If the student rejects, then both student and tutor earn zero.) Note: the tutor may offer the guarantee, but is not required to do so. Assume that the student cannot deliberately fail the examination.

Suppose that a tutor asks x and offers a double-your-money-back guarantee. The student's expected payoff is $p(V - x) + (1 - p)x$. Hence the student will accept prices as large as $\frac{3V}{2}$ as long as they are guaranteed (these prices lead to non-negative surplus for any $p \in [\frac{1}{4}, \frac{3}{4}]$). As in the previous part, a non-guaranteed price of $\frac{V}{4}$ or less will always be accepted. Note that if $p = \frac{3}{4}$, then the guaranteed price of $\frac{3V}{2}$ earns $\frac{3V}{4}$, which is the most that the good tutor can earn in any equilibrium and strictly more than in any pooling equilibrium (because he is getting all of the student's surplus); while if $p = \frac{1}{4}$, then this price earns $-\frac{3V}{4}$. It follows that one equilibrium is for the good tutor to set $x = \frac{3V}{2}$ and offer the guarantee and for the bad tutor to set $x = \frac{V}{4}$ without the guarantee. The student should accept unguaranteed prices if and only if they are lower than $\frac{V}{4}$ and any guaranteed price no greater than $\frac{3V}{2}$ (but no larger). Supporting beliefs: high-quality if and only if a guarantee is offered, independent of price. This argument establishes that the high-quality tutor can guarantee utility $\frac{3V}{4}$ in equilibrium. This cannot be done without offering a guarantee (because the low-quality tutor would also make the offer and then it would not be optimal for the student to accept it). Consequently, the high-quality tutor must ask $\frac{3V}{2}$ with a guarantee in equilibrium. This offer makes negative profit for the low-quality tutor. Therefore he must offer a separate price. The unguaranteed price of $\frac{V}{4}$ is the best that he can do. Hence I have described the unique equilibrium outcome.

3. Consider a labor market in which there is one worker and two potential employers. The worker is described by one characteristic, height. Height h can take on one of two values, tall ($h = 1$) or short ($h = 2$). Each worker has a reservation utility r_h that depends height. Assume that $r_h = \alpha_h$, where $0 < \alpha_1 < \alpha_2 < 1$. If a firm employs the worker at wage w, the firm earns $h - w$ when the worker's height is h. If a firm does not employ the worker, the firm earns 0. If a worker accepts a job with wage w, the worker earns w. If the worker does not accept a job, the worker earns r_h. The game works like this. First, nature picks h. Second, the worker learns h. Third, Firm 1, which conducts in-person interviews, learns h. Firm 2 does not learn h. Fourth, Firms 1 and 2 simultaneously make wage offers to the worker. Fifth, the worker selects one of the job offers or rejects both. Firms 1 and 2 are different, therefore, because Firm 1 can condition its offer on the worker's height and Firm 2 cannot.

   (a) If Firm 1 offers 0 wages (to both tall and short workers), what is Firm 2's best response (assuming that the worker responds by maximizing her payoffs).

   If Firm 2 offers p then it will attract both types if $p \geq \alpha_2$. The best such strategy would be to set $p = \alpha_2$. FIrm 2 would always buy something. The average quality of the purchase would be $2q + (1 - q)$ where q is the probability that the worker is short. Alternatively, Firm 2 can offer $p \in [\alpha_1, \alpha_2)$ and sell only to $h = 1$ and, when it does so, earn $1 - p$. Hence the firm will offer α_1 if $(1-q)(1-\alpha_1) \geq 1+q-\alpha_2$ and α_2 otherwise (being indifferent when $(1-q)(1-\alpha_1) = 1+q-\alpha_2$).

   (b) If Firm 2 offers the wage 0, what is Firm 1's best response (assuming that the worker responds by maximizing her payoffs)?

   Firm 1 can discriminate. It will offer α_i to workers of type i.

   (c) Prove that, in any subgame perfect equilibrium, Firm 2 makes zero profits.

   Firm 2 charges the same price to all types. If Firm 2 makes positive profits, then it would certainly make positive profits on sales to $h = 2$ types. Firm 1 can offer a price exclusively to these types. If Firm 2 earns positive profits, then Firm 1 can increase its profits by offering a wage to short workers that is slightly above Firm 1's wage. (This answer assumes that Firm 2 is playing a pure strategy.)

   (d) Prove that, in any subgame perfect equilibrium, Firm 2 never hires short workers.

   If Firm 2 were to hire short workers, then tall workers would always accept the offer. (Firm 1 would not offer the same wage to high and low types in equilibrium unless it was selling with probability one.) it would therefore necessarily make positive profits on the short workers (or negative profit overall). But then the first firm would undercut Firm 2's wage (to short workers). (This answer assumes that Firm 2 is playing a pure strategy.)

   (e) Describe the subgame perfect equilibria of the game.

   Assuming that Firm 2 plays a pure strategy, Firm 2 must make zero profit and sell only to low types. Hence Firm 2 pays no more than 1 and if it pays less than 1 it hires no one. It cannot be an equilibrium for Firm 2 to pay less than 1: One of the firms would have the incentive to offer a slightly higher wage and attract all workers at a profitable wage. If Firm 2 pays 1, then Firm 1 would never pay more than 1. This is the only possible pure strategy equilibrium. But it cannot be an equilibrium because given Firm 1's behavior, Firm 2 can profit by offering slightly more than 1 (and hiring both short and tall workers). This means that no pure-strategy equilibrium exists.

   There is a mixed equilibrium. In the mixed equilibrium, Firm 2 always offers 1 to tall workers. Firm 2 uses the cdf F_2 to describe its wage offers to short workers: $F_2(w)$ is the probability that

Firm 2's offer to short workers is less than or equal to 2. Firm 1's cdf is F_1. To be an equilibrium Firms must be indifferent between all wages in the support. If p_h is the probability of type h and Firm 1 offers w to $h = 2$, then it earns (in expectation, conditional on the worker being $h = 2$):

$$(2 - w)F_2(w).$$

Interpretation: Firm 1 only hires $h = 2$ workers. When it hires such a worker, the firm earns $2 - w$. It hires the worker if Firm 2 offers less than w, which is $F_2(w)$ (strictly speaking I am assuming that there is no "atom" of probability in Firm 2's distribution at w).

Hence $(2 - w)F_2(w) = K_1$ for some constant K_1 and for w in the support of F_1. Similarly, Firm 1 earns:

$$(2 - w)F_1(w)p_2 + (1 - w)p_1$$

where the first term is what Firm 1 makes when it hires a short worker and the second term what it makes when it hires a tall worker. Notice that (assuming $w > 1$) Firm 1 always succeeds in hiring the tall worker but only hires the short worker if Firm 1's wage was low. Again, there expected profits must be constant throughout the support:

$$(2 - w)F_1(w)p_2 + (1 - w)p_1 = K_2.$$

The next step is to identify the support of the distributions.

The highest price in the support of the two distributions must be the same (otherwise one firm can increase profit by never bidding above the other firm's support). The highest price Firm 2 offers cannot be greater than $E(h)$ (else the firm makes negative profits because it always hires low types). If the common upper bound is strictly less than $E(h)$, both firms make positive expected profits (because offering the highest wage attracts the worker with probability one). The lowest wage that FIrm 2 offers cannot attract $h = 2$ workers (else Firm 1 is offering too little). Hence Firm 2's lowest wage must be less than or equal to 1, but Firm 2 will bid the wage for $h = 1$ workers up to one. It follows that the lowest wage in the support of both distributions is 1 and that, therefore, Firm 2 earns zero expected profit. This means that $K_2 = 0$ and

$$F_1(w) = \frac{(w - 1)p_1}{(2 - w)p_2}.$$

Further, $F_2(p_1 + 2p_2) = 1$ since Firm 2 never offers more than expected productivity, $(2 - w)F_2(w) = K_1$ implies that $K_1 = 2 - p_1 - 2p_2 = p_1$ and so

$$F_2(w) = \frac{p_1}{2 - w}.$$

These strategies do constitute an equilibrium. Note that F_2 has an atom at $w = 1$.