Simple Math:
Solutions to Bayes's Rule Problems

Comments. The first two questions were too hard for everyone. Most people arrived at the correct answer for the third question. Results were mixed on the fourth and the fifth questions. For question four, several people stated that the probability of your original choice goes up after Monty opens three doors. I do not understand why this should be true if you know in advance that Monty will always open three empty doors. If the probability of your choice doesn't change, then the "remaining" probability $\frac{4}{7}$ must all be behind the door Monty didn't open. I wanted people to observe how the fifth question was different from the Monty Hall problem. Some people did. Some people did not. Problem 6 was poorly worded. I should have said: Suppose you do not know whether Monty always opens an empty door and you suspect that he does not want you to win a prize. When it is your turn to play the game, you open Door #1, Monty opens Door #2, and then he offers you the chance to open Door #3. What do you do?

Exercise 1 In front of you is a bookbag containing 1,000 poker chips. I started out with two such bookbags, one containing 700 red and 300 blue chips, the other containing 300 red and 700 blue. I flipped a fair coin to determine which bookbag to use, so your prior probability that the bookbag in front of you is the red bookbag is 50%. Now, you sample randomly, with replacement after each chip. In 12 samples, you get 8 reds and 4 blues. What is the probability that this is the predominantly red bag?

Solution 1 This is how I originally answered the question: According to a study performed by Lawrence Phillips and Ward Edwards in 1966, most people, faced with this problem, give an answer in the range 70% to 80%. Did you give a substantially higher probability than that? If you did, congratulations. Ward Edwards wrote that very seldom does a person answer this question properly, even if the person is relatively familiar with Bayesian reasoning. The correct answer is 97%.

A blue chip is exactly the same amount of evidence as a red chip, just in the other direction. If you draw one blue chip and one red chip, they cancel out. So the ratio of red chips to blue chips does not matter; only the excess of red chips over blue chips matters. There were eight red chips and four blue chips in twelve samples; therefore, four more red chips than blue chips. Computation simplifies to $\frac{.7^4}{.7^4+.3^4} = 97\%$.

Now that I have looked at your answers, I see that you had a lot of trouble. So here is a more detailed approach.

Bayes's Rule says:

$$P(R = r \mid e) = \frac{P(e \mid R = r)P(R = r)}{\pi(e)}.$$

We need to figure out what everything stands for in the first problem. e is the outcome of the experiment. The experiment is to take 12 samples; the outcome is to obtain 8 reds and 4 blues. $\pi(e)$ is the probability that the outcome of the experiment is e. R is the characteristic of the bag. $R =$red means that the contents of the bag is mostly (700) red chips. $R =$ blue means that the contents of the bag is mostly blue. $P(R = r)$ is the probability that the contents of the bag is "r" (which can stand for either red or blue). Finally, $P(e \mid R = r)$ is the probability that the outcome of the experiment is e if the contents of the bag is r (so, for example, for us e is always drawing 8 reds out of 12, so $P(e \mid R = \text{red})$ is the probability that the you'll draw 8 reds out of 12 when the bag is mostly red. Finally, what we want to find is $P(R = \text{r} \mid e)$, the probability that the bag contains mostly red if the result of the experiment is e.

Now that we know what the terms are, all that is left is to figure out the values for the probabilities on the right-hand side and use Bayes's Rule to figure out the probability on the left. The easy thing is $P(R = r)$ this is one half (whether r is red or blue). I know this because the problem says "I flipped a fair coin and ...." Second, $P(e \mid R = r) = \binom{12}{4}(.3)^4(.7)^8$ when r is red and $P(e \mid R = r) = \binom{12}{4}(.3)^8(.7)^4$ when r is blue. Maybe you don't know these formulas. I'll explain them below. For now, just assume that they are true. All that is left is to compute $\pi(e)$, but we have a general formula for that:

$$\pi(e) = P(R = \text{ blue})P(e \mid R = \text{ blue}) + P(R = \text{ red})P(e \mid R = \text{ red}).$$

Now, plug everything into the original formula, cancelling similar terms to obtain:

$$P(R = \text{ red} \mid e) = \frac{.7^4}{.7^4 + .3^4} = .97.$$

About the formula for $P(e \mid R = r)$: What you are doing is computing the probability that you get 8 reds out of 12 when the probability of getting a red on any one draw is .7. Let's ask a different question. What is the probability that the first 4 chips you pick are blue and the next eight are red? The answer to this should be clear: $(.3)^4(.7)^8$ (the first four things happen with probability .3 each; the last eight with probability .7 each). Now this is just one way of may in which you can pull out exactly four blues. The total probability is obtained by multiplying $(.3)^4(.7)^8$ by the number of subsets of size four there is in a group of 12. This is just the binomial coefficient $\binom{12}{4}$. ($12 \times 11 \times 10 \times 9$ is the number of different ordered groups of four you can pick out of twelve things: there are 12 candidates for the first choice; 11 for the second; and so on .... $4 \times 3 \times 2 \times 1$ is the number of ways you can arrange four things, so you divide this into the first number to obtain the number of (unordered) groups of 4.)

The fact that there is so much cancellation in the Bayes's Rule expression suggests that there is an easier way to arrive at the conclusion. There is. Since the problem is symmetric, a "balanced" sample (same number of reds as blues) tells you nothing new about which color dominates. So getting 8 red out of twelve is exactly as informative as getting 4 red out of 4, which happens with probability $.7^4$ if the bag is mostly red and $.3^4$ if it is mostly blue. This observation explains why the answer to the second question is the same as the answer to the first one.

Exercise 2 How does the answer to the previous question change if sixteen chips were sampled and we found ten red chips and six blue chips.

Solution 2 The bookbag problem would have exactly the same answer, obtained in just the same way, if R red chips and B blue chips were drawn and $R = B + 4$.

Exercise 3 You are a mechanic for gizmos. When a gizmo stops working, it is due to a blocked hose 30% of the time. If a gizmo's hose is blocked, there is a 45% probability that prodding the gizmo will produce sparks. If a gizmo's hose is unblocked, there is only a 5% chance that prodding the gizmo will produce sparks. A customer brings you a malfunctioning gizmo. You prod the gizmo and find that it produces sparks. What is the probability that a spark-producing gizmo has a blocked hose?

Solution 3

$$\frac{.3(.45)}{.3(.45) + .7(.05)} = 79\%$$

Exercise 4 Suppose there are seven doors. You choose three doors. Monty now opens three of the remaining doors to show you that there is no prize behind any of them. He then says, "Would you like to stick with the three doors you have chosen, or would you prefer to swap them for the one other door I have not opened?" What do you do? Do you stick with your three doors or do you make the 3 for 1 swap he is offering?

Solution 4 Again assume that the prize is initially equally likely to be behind any of the seven doors and that Monty knows where the prize is; always opens three doors that do not have the prize; and, when he has a choice, randomly selects a door to open.

Here you initially had a probability of $\frac{3}{7}$ of winning and the choice will give you a probability of $\frac{4}{7}$, so you should switch. Following the formulas, assume that you pick the first three doors. The probability that Monty opens a particular group of the other doors, say $\{5, 6, 7\}$, $p(\{5, 6, 7\})$ is

$$p(\{5,6,7\}) = \pi(4)(1) + \pi(\{5,6,7\})(0) + \pi(\{1,2,3\})(\frac{1}{4}) = \frac{1}{7} + \frac{3}{7}\frac{1}{4} = \frac{1}{4}.$$

$$P(\#4 \mid \{5,6,7\}) = \frac{\frac{1}{7}}{\frac{1}{4}} = \frac{4}{7}.$$

Exercise 5 Suppose you are taking an examination. The first question is a multiple choice question with three choices. Only one of them is correct. You don't know the answer and randomly select the first choice. At this point, the professor (who has not seen your answer, but knows the correct answer to the question) announces: "It looks like the exam is quite hard. Here is a hint: The second choice for the answer to Question 1 is wrong. Please ignore this option and select the correct answer from either the first or the third choice." Is it right to change your answer?

Solution 5 Here the professor is not quite like Monty Hall because she does not know the choice you made. Presumably some students selected the second choice. The information provided by the announcement tells you something about the accuracy of your first choice. It is reasonable to assume that the professor selected an alternative to delete at random. If so, the probability that she deletes the second choice is $\frac{1}{3}$; the probability she deletes the second choice given that the first choice is correct is one half; and the probability that the first choice is correct is one third. The appropriate Bayes's Rule expression is therefore:

$$P(1 \mid 2) = \frac{P(2 \mid 1)\pi(1)}{p(2)} = \frac{(\frac{1}{2})(\frac{1}{3})}{\frac{1}{3}} = \frac{1}{2}.$$

Hence the information raises the chances that you guessed the correct answer from $\frac{1}{3}$ to $\frac{1}{2}$. It also raises the chance that the third choice is correct from $\frac{1}{3}$ to $\frac{1}{2}$. So you do not gain anything from changing (but your expected score on the exam goes up). On the other hand if you had initially guessed the second choice, you would plainly switch.

Exercise 6 Suppose Monty Hall has a choice about whether to open another door or whether to offer you a chance to change. How would you play the game?

Solution 6 Properly formulated, this is a problem in "zero-sum game theory." You expect to win with probability $\frac{1}{3}$ if Monty offers you no information and no choices. Monty can guarantee that you do no better than this. These observations suggest that you cannot expect to gain from accepting any offer that Monty makes. A complete analysis involves making some assumptions to clarify just what Monty can do and what he is trying to achieve. I raise the possibility because it is a natural one. It is reasonable to assume that Monty has his own agenda. Different ways of interpreting Monty's objectives lead to different answers. Certainly if Monty only offers a choice when the prize is behind Door # 1, you should never change when offered a change.