Linear Programming Notes V
Problem Transformations

1 Introduction

Any linear programming problem can be rewritten in either of two standard forms. In the first form, the objective is to maximize, the material constraints are all of the form: "linear expression $\leq$ constant" ($a_i \cdot x \leq b_i$), and all variables are constrained to be non-negative. In symbols, this form is:

$$\max c \cdot x \text{ subject to } Ax \leq b, x \geq 0.$$

In the second form, the objective is to maximize, the material constraints are all of the form: "linear expression = constant" ($a_i \cdot x = b_i$), and all variables are constrained to be non-negative. In symbols, this form is:

$$\max c \cdot x \text{ subject to } Ax = b, x \geq 0.$$

In this formulation (but not the first) we can take $b \geq 0$.

Note: The c, A, b in the first problem are not the same as the c, A, b in the second problem.

In order to rewrite the problem, I need to introduce a small number of transformations. I'll explain them in these notes.

2 Equivalent Representations

When I say that I can rewrite a linear programming problem, I mean that I can find a representation that contains exactly the same information. For example, the expression $2x = 8$ is equivalent to the expression $x = 4$. They both describe the same fact about x. In general, an equivalent representation of a linear programming problem will be one that contains exactly the same information as the original problem. Solving one will immediately give you the solution to the other.

When I claim that I can write any linear programming problem in a standard form, I need to demonstrate that I can make several kinds of transformation: change a minimization problem to a maximization problem; replace a constraint of the form ($a_i \cdot x \leq b_i$) by an equation or equations; replace a constraint of the form ($a_i \cdot x \geq b_i$) by an equation or equations; replace a constraint of the form ($a_i \cdot x = b_i$) by an inequality or inequalities; replace a variable that is not explicitly constrained to be non-negative by a variable or variables so constrained. If I can do all that, then I can write any problem in either of the desired forms.

3 Rewriting the Objective Function

The objective will be either to maximize or to minimize. If you start with a maximization problem, then there is nothing to change. If you start with a minimization problem, say min $f(x)$ subject to $x \in S$, then an equivalent maximization problem is max $-f(x)$ subject to $x \in S$. That is, minimizing $-f$ is the same as maximizing f. This trick is completely general (that is, it is not limited to LPs). Any solution to the minimization problem will be a solution to the maximization problem and conversely. (Note that the value of the maximization problem will be -1 times the value of the minimization problem.)

In summary: to change a max problem to a min problem, just multiply the objective function by -1.

4 Rewriting a constraint of the form $(a_i \cdot x \leq b_i)$

To transform this constraint into an equation, add a non-negative slack variable:

$$a_i \cdot x \leq b_i$$

is equivalent to

$$a_i \cdot x + s_i = b_i \text{ and } s_i \geq 0.$$

We have seen this trick before. If x satisfies the inequality, then $s_i = b_i - a_i \cdot x \geq 0$. Conversely, if x and s_i satisfy the expressions in the second line, then the first line must be true. Hence the two expressions are equivalent. Note that by multiplying both sides of the expression $a_i \cdot x + s_i = b_i$ by -1 we can guarantee that the right-hand side is non-negative.

5 Rewriting a constraint of the form $(a_i \cdot x \geq b_i)$

To transform this constraint into an equation, subtract a non-negative surplus variable:

$$a_i \cdot x \geq b_i$$

is equivalent to

$$a_i \cdot x - s_i = b_i \text{ and } s_i \geq 0.$$

The reasoning is exactly like the case of the slack variable.

To transform this constraint into an inequality pointed in the other direction, multiply both sides by -1.

$$a_i \cdot x \geq b_i$$

is equivalent to

$$-a_i \cdot x + s_i \leq -b_i.$$

6 Rewriting a constraint of the form $(a_i \cdot x = b_i)$

To transform an equation into inequalities, note that $w = z$ is exactly that same as $w \geq z$ and $w \leq z$. That is, the one way for two numbers to be equal is for one to be both less than or equal to and greater than or equal to the other. It follows that

$$a_i \cdot x = b_i$$

is equivalent to

$$a_i \cdot x \leq b_i \text{ and } a_i \cdot x \geq b_i.$$

By the last section, the second line is equivalent to:

$$a_i \cdot x \leq b_i \text{ and } -a_i \cdot x \leq -b_i.$$

7 Guaranteeing that All Variables are Explicitly Constrained to be Non-Negative

Most of the problems that we look at requires the variables to be non-negative. The constraint arises naturally in many applications, but it is not essential. The standard way of writing linear programming problems imposes this condition. The section shows that there is no loss in generality in imposing the restriction. That is, if you are thinking about a linear programming problem, then I can think of a mathematically equivalent problem in which all of the variables must be non-negative.

The transformation uses a simple trick. You replace an unconstrained variable x_j by two variables u_j and v_j. Whenever you see x_j in the problem, you replace it with $u_j - v_j$. Furthermore, you impose the constraint that $u_j, v_j \geq 0$. When you carry out the substitution, you replace x_j by non-negative variables. You don't change the problem. Any value that x_j can take, can be expressed as a difference (in fact, there are infinitely many ways to express it). Specifically, if $x_j \geq 0$, then you can let $u_j = x_j$ and $v_j = 0$; if $x_j < 0$, then you can let $u_j = 0$ and $v_j = -x_j$.

8 What Is the Point?

The previous sections simply introduce accounting tricks. There is no substance to the transformations. If you put the tricks together, they support the claim that I made in the beginning of the notes. Who cares? The form of the problem with equality constraints and non-negative variables is the form that the simplex algorithm uses. The inequality constraint form (with non-negative variables) is the form used in for the duality theorem.

Warnings: These transformations really are transformations. If you start with a problem in which x_1 is not constrained to be non-negative, but act as if it is so constrained, then you may not get the right answer (you'll be wrong if the solution requires that x_1 take a negative value). If you treat an equality constraint like an inequality constraint, then you'll get the wrong answer (unless the constraint binds at the solution). Similarly, you can't treat as inequality constraint as an equation in general. The transformations involve creating a new variable or constraint to compensate for the changing inequalities to equations, equations to inequalities, or whatever it is you do.

9 Example

You know all the ideas. Let me show you how they work. Start with the problem:

$$\begin{array}{lrcrcrcr} \text{min} & 4x_1 & + & x_2 & & & & \\ \text{subject to} & -2x_1 & + & x_2 & & & \geq & 6 \\ & & & x_2 & + & x_3 & = & 4 \\ & x_1 & & & & & \geq & -4 \\ & & & x_2 & & x_3 & \geq & 0 \\ . & & & & & & & \end{array}$$

and let's write it in either of the two standard forms.

First, to get it into the form:

$$\max c \cdot x \text{ subject to } Ax \leq b, x \geq 0$$

change the objective to maximize by multiplying by -1:

$$\max -4x_1 - x_2.$$

Next, change the constraints. Multiply the first constraint by -1:

$$2x_1 - x_2 \leq -6;$$

replace the second constraint by two inequalities:

$$x_2 + x_3 \leq 4 \text{ and } -x_2 - x_3 \leq -4;$$

and replace the third constraint by the inequality:

$$-x_1 \leq 4.$$

Finally, replace the unconstrained variable x_1 everywhere by $u_1 - v_1$ and add the constraints that $u_1, v_1 \geq 0$. Putting these together leads to the reformulated problem

$$
\begin{array}{lrcrcrcrcr}
\text{max} & -4u_1 & + & 4v_1 & - & x_2 & & & & \\
\text{subject to} & 2u_1 & - & 2v_1 & - & x_2 & & & \le & -6 \\
& & & & & x_2 & + & x_3 & \le & 4 \\
& & & & - & x_2 & - & x_3 & \le & -4 \\
& -u_1 & + & v_1 & & & & & \le & 4 \\
& u_1 & & v_1 & & x_2 & & x_3 & \ge & 0
\end{array}
$$

.

In notation, this problem is in the form: $\max c \cdot x$ subject to $Ax \le b, x \ge 0$ with $c = (-4, 4, -1, 0), b = (-6, 4, -4, 4)$ and

$$
A = \begin{bmatrix} 2 & -2 & -1 & 0 \\ 0 & 0 & 1 & 1 \\ 0 & 0 & -1 & -1 \\ -1 & 1 & 0 & 0 \end{bmatrix}.
$$

Next, to put the problem into the form: $\max c \cdot x$ subject to $Ax = b, x \ge 0$, change the objective function to max as above; replace x_1 by $u_1 - v_1$ as above; replace the first constraint with

$$-2u_1 + 2v_1 + x_2 - s_1 = 6 \text{ and } s_1 \ge 0;$$

leave the second constraint alone; and replace the third constraint with

$$-u_1 + v_1 + s_3 = 4.$$

The problem then becomes

$$
\begin{array}{lrcrcrcrcrcrcr}
\text{max} & -4u_1 & + & 4v_1 & - & x_2 & & & & & & & & \\
\text{subject to} & -2u_1 & + & 2v_1 & + & x_2 & & & - & s_1 & & & = & 6 \\
& & & & & x_2 & + & x_3 & & & & & = & 4 \\
& -u_1 & + & v_1 & & & & & & & + & s_3 & = & 4 \\
& u_1 & & v_1 & & x_2 & & x_3 & & s_1 & & s_3 & \ge & 0
\end{array}
$$

.

In notation, this problem is in the form: $\max c \cdot x$ subject to $Ax = b, x \ge 0$ with $c = (-4, 4, -1, 0, 0, 0), b = (6, 4, 4)$ and

$$
A = \begin{bmatrix} -2 & 2 & 1 & 0 & -1 & 0 \\ 0 & 0 & 1 & 1 & 0 & 0 \\ -1 & 1 & 0 & 0 & 0 & 1 \end{bmatrix}.
$$

In the two different transformations, the A, b, and c used in the representation differ. Indeed, the two descriptions of the problem have different numbers of variables and different numbers of constraints. It does not matter. If you solve either problem, you can substitute back to find values for the original variables, $x_1, \ldots, x_4$, and the original objective function.

Computer programs that solve linear programming problems (like Excel) are smart enough to perform these transformations automatically. That is, you need not perform any transformations of this sort in order to enter an LP into Excel. The program asks you whether you are minimizing or maximizing, whether each constraint is an inequality or an equation, and whether the variables are constrained to be nonnegative.