Linear Programming Notes IX: Two-Person Zero-Sum Game Theory

1 Introduction

Economists use the word rational in a narrow way. To an economist, a rational actor is someone who makes decisions that maximize her (or his) preferences subject to constraints imposed by the environment. So, this actor knows her preferences and knows how to go about optimizing. It is a powerful approach, but it probably is only distantly related to what you mean when you think of yourself as rational.

Decision theory describes the behavior of a rational actor when her actions do not influence the behavior of the people around her. Game theory describes the behavior of a rational actor in a strategic situation. Here decisions of other actors determine how well you do. Deciding where to go to dinner can be thought of as a decision problem if all you care about is what you eat and where you eat it. It is a strategic problem if you also want to meet a friend at the restaurant. (In the first case, you go to the restaurant that serves the food you like best. In the second case, the restaurant that you prefer depends not only on the food served, but also on the where your friend goes.)

2 Zero-Sum Games

These notes describe a simple class of games called two-player zero-sum games. You can probably figure out what a two-player game is. Zero-sum games refer to games of pure conflict. The payoff of one player is the negative of the payoff of the other player. This formulation is probably appropriate for most parlor games, where the outcomes are either win, lose, or draw (and there is at most one winner or loser). Maybe it describes war. It is a restrictive assumption and is not appropriate to most economic applications, where there is a strong component of common interests mixed with the conflict. For example, in a bargaining situation, the conflict is clear: the buyer wants to pay a low price and the seller wants to receive a high price. The cooperative element arises because it is frequently the case that making a transaction at an intermediate price is better for both sides than a failure to reach an agreement. Concretely, if something is worth $10 to the seller and $15 to the (potential) buyer, then making a sale at the price $12 (or any price between $10 and $15) is better for both buyer and seller than making no sale at all. Problems that describe aspects of firm competition (models of Cournot duopoly that you may have seen in a micro class) have non-zero sum aspects.

Why limit attention to zero-sum games? They are simpler. There is a beautiful theory that is more compelling than the general theory of games. Predicting outcomes in these games uses linear programming in ways that do not generalize to other kinds of game.

The general structure of a game involves a list of players; a set of strategies for each of the players; a payoff for each vector of strategies. I will assume that the game has only two players.

3 Strategies

The intuition behind a strategy is that it tells you how you are going to play the game. In examples, it will be just a choice from one of a finite list of possible things you can do.

This story might help you understand the notion of a strategy. You made an arrangement to talk to a friend about what you were going to do together, but you unexpectedly cannot be home when the friend is supposed to call. Your roommate will be home and promises to talk to your friend. You want to give your roommate instructions about what kind of arrangements to make. You would like to walk on the beach, but not if it is going to rain. You would like to go to the Belly Up, but only if you can dance. You would like to see a movie, but only if Leonardo DiCaprio isn't in it. Most of all, you would like to do something that your friend also wants to do. What kind of instructions do you give your roommate? Complete instructions will account for all possible contingencies. You won't say: "Tell my friend that I'll do whatever he or she wants to do." Instead, you'll say something like: "If she wants to go to a movie, find out if DiCaprio is in it. If he isn't, tell her OK. If he is, tell her no." And so on. In game theory, a strategy is a complete set of instructions. It allows your roommate to "negotiate" for you no matter what your friend on the phone says.

When you specify a strategy for each player, you determine the **outcome** of the game. Payoffs associate to each outcome a number for each player. You can therefore describe two-player games using a payoff matrix. The rows of the matrix represent the strategies of one player. The columns of the matrix represent the strategies of the other player. The cells of the matrix represent outcomes. In these cells, you place payoff numbers. In general, each cell should have a payoff for each player in it. In zero-sum games, you need only have one number in each cell. This number represents the payoff to the player who picks rows. The negative of this number is the payoff to the player who picks columns.

Take the game of matching pennies. Two players simultaneously place a penny on the table. If the pennies 'match' (both heads up or both heads down), then the Row player wins the Column player's penny. If the pennies do not 'match' (exactly one head), then the Column player wins the Row player's penny. The payoff matrix is below.

	Heads	Tails
Heads	1	-1
Tails	-1	1

In matching pennies, each player has two strategies. The player can either play heads or play tails. Now consider a variant of matching pennies that I play with my son. First, I decide whether to play heads or tails. Next, he looks at what I did. Finally, he decides whether to play heads or tails. I win if the coins match. He wins if they do not. In this game, both players must decide whether to play heads or tails. So you might think that we both have two strategies. This is not correct. I have two strategies, but my son can make his decision based on what I did. He therefore has four strategies:

HH: Play heads no matter what I do.

TT: Play tails no matter what I do.

HT: Play heads if I play heads and tails if I play tails (match).

TH: Play tails if I play heads and heads if I play tails (mismatch).

Therefore the payoff matrix for this version of matching pennies is:

	HH	TT	HT	TH
Heads	1	-1	1	-1
Tails	-1	1	1	-1

Naturally, my son plays TH and I always lose. The point is that even though my son ends up either playing heads or playing tails, in order to describe how he makes his decision, you need four strategies. Using the four strategies he could give instructions to my wife on how to play the game, go to his room and listen to music, and still always win the game.

Strategies are complicated objects in general. Examples simplify and obscure the complexity of the idea of a strategy. For example, chess is a zero-sum, two-player game. A strategy for chess (to a game theorist) is a complete plan for playing that game. If you are white (and move first), your strategy should include an opening move, a response to *all possible* first moves of you opponent; a response to *all possible* positions after two moves by your opponent; and so on. There are an enormous number of such strategies (no, not on the order of the number of pennies in Bill Gates's bank account; more like the number of water molecules in the universe). The idea is that if you could specify a strategy, then you can tell the strategy to an agent and the agent will be able to play the game for you without ever consulting you again. Once you have a strategy for both white and black, you can actually play out a game. From the play of the game, you can decide who won (or whether it was a draw) and assign payoffs. Conceptually, this process is easy (at least for someone who is comfortable with game theory). In practice, it does not tell you how to play a game. Tic-tac-toe is a simpler example of a two-player zero-sum game. To a game theorist, a strategy for the first player describes the first move and where to move on future opportunities under all possible circumstance. This leads to an enormous number of strategies. You have been able to play tic-tac-toe optimally for more than fifteen years. You can probably even describe it (move in the center first; after that block your opponent when necessary; move to an open corner if you can). Here the point is that describing all of the strategies even for tic-tac-toe is an enormous task and it is not directly related to what you think about when you actually play the game.

Game theory does provide advice about how to play simple zero-sum games. The first advice is about which strategies to avoid. In the payoff matrix below, Row picker always does better picking UP than DOWN. That is, the entries in each column of row one are bigger than the corresponding numbers in the second row. No matter what Column player two selects, player one is better off picking Row 1 than Row 2. If Row wants to maximize his return, he will avoid the DOWN row. We say that DOWN is a dominated strategy.

	A	B	C	D
UP	1	2	3	4
DOWN	-1	-2	-3	-4

4 Examples

In this section I will describe some fairly simple games. The goal is to use the notion of a strategy to describe the games. After I have presented the theory, we will return to the games.

4.1 Colonel Blotto

Several standard examples of games have charming names like "The No-Left Turn Missle" and "Search and Destroy." These names suggest that hot and cold warriors used game theory to think about military strategy. They did. This is a simple example of a class of games that describe some aspect of military strategy.

Colonel Blotto has three divisions to defend two mountain passes. He will defend successfully against equal or smaller strength, but lose against superior forces. The enemy has two divisions. The battle is lost if either pass is captured. Neither side has advance information on the disposition of the opponent's divisions. What are the optimal dispositions?

Colonel Blotto has to decide how many divisions to allocate to the first mountain pass (he'll allocate the remaining ones to the other pass). You can describe a strategy with a pair of numbers like $(x, 3-x)$, where $x = 0, 1, 2$, or 3. x represents the troops allocated to the first pass; $3-x$ the troops allocated to the second path. Similarly, the enemy's strategy is a pair, but since it has only two divisions, it has only three strategies. Hence I obtain the payoff matrix below.

	(2,0)	(1,1)	(0,2)
(3,0)	1	-1	-1
(2,1)	1	1	-1
(1,2)	-1	1	1
(0,3)	-1	-1	1

Consider the first row. Colonel Blotto allocates three divisions to the first pass. Therefore he always defeats the enemy there, but he only defeats the enemy on the second pass when the enemy also allocates all of its troops to the left pass. Since Blotto loses the war unless he can defend both passes, his payoff is negative one when the enemy uses either $(1, 1)$ and $(0, 2)$. If Blotto allocates two units to the first pass (the second row), then he successfully defends the first pass and win also defend the second pass if the enemy allocates fewer than two divisions to the second pass. Hence Blotto wins unless his enemy plays $(0, 2)$. Similar reasoning explains the rest of the table.

4.2 Morra

Each player shows either one or two fingers and announces a number between 2 and 4. If a player's number is equal to the sum of the number of fingers shown, then his opponent must pay him that many dollars. The payoff is the net transfer (so that both players earn zero if both or neither guess the correct number of fingers shown).

In this game each player has 6 strategies: he may show one finger and guess 2; he may show one finger and guess 3; he may show one finger and guess 4; or he may show two fingers and guess one of the three numbers. Of these 6 strategies, two are stupid and I will ignore them. It never pays to put out one finger and guess that the total number of fingers will be 4 (because the other player can put out more than two fingers). It never pays to put out two fingers and guess that the sum will be 2 (because the other player must put down at least one finger). Therefore, a four by four matrix describes the payoffs.

	12	13	23	24
12	0	2	-3	0
13	-2	0	0	3
23	3	0	0	-4
24	0	-3	4	0

In the payoff matrix, "12" describes the strategy of putting out one finger and guessing the sum is two. In general, the first number in the strategy is the number of fingers and the second number is the (guessed) sum. The payoffs come from playing out the game. Suppose that both players use 12. Then both put out one finger. The sum is equal to two. So each player pays \$2 to his opponent. They break even. This explains why the payoff associated with both players playing 12 is equal to zero. Moving to the second entry in the first row (Row plays 12; Column plays 13): here both players put out one finger; the row player correctly guesses the sum is 2; Column must pay Row the this amount. When Row plays 12 and Column 23, the sum is 3; Column guesses it correctly (but Row's guess is incorrect); so Column receives \$3 from Row.

4.3 Goofspiel

Each player begins with an $n-$card "hand," with cards numbered $1, 2, \ldots, n$. On the first move of the game, each player picks a card from his hand. The cards are compared. The player with the higher card earns a_1 dollars. The player with the lower card earns 0. If the cards are equal, then each player wins $\frac{a_1}{2}$. On the next move, each player picks one of the cards remaining in his hand. As before, the cards are compared. The player who put out the higher card earns a_2 dollars; the player with the lower card earns 0. If the cards are equal, then each player wins $\frac{a_1}{2}$. The play continues until all n cards have been played. The possible winnings in the ith round is a_i. The total payoff is equal to the sum of the winnings in the individual moves.

While it did not take long to describe this game, the strategy space is enormous. A player does not just select an order to play his cards (and there are $n!$ possible orders). Instead a strategy allows the player to decide which card to play on the basis of what his opponent has done. Only when $n = 2$ is the strategy set small. Here the player really need only decide what to play on his first move. On the second move he must play his remaining card. When there are three cards, there are 24 strategies. You can describe them as a list. The list contains which card you play first (3 possible choices); what card you play second if your opponent plays 1 in the first round (2 possible choices because you have already played a card); what card you play second if your opponent plays 2 in the first round (2 possible choices); and what card you play second if your opponent plays 3 in the first round (2 possible choices). Hence each player has $3 \times 2 \times 2 \times 2 = 24$ possible strategies. Observe the complexity of the notion of strategy. Your opponent is going to play one card on his first move. Nevertheless, your strategy describes how you respond to all potential first moves. The reason for this complexity is that you pick you strategy in advance. That is, a strategy will typically specify how you would behave in contingencies that do not actually take place. A tiny aspect of the strategy is simple. Once the strategy describes how the first two cards are played, it does not need to say anything about the third card. On the third move (when $n = 3$) a player must play his one remaining card.

5 Security Level

Imagine now that your opponent can read your mind and guess how you play before she makes her move. What should you do? Since your opponent gains when you lose, you should expect your opponent to pick the strategy that makes you worse off. Take a look at the next example.

	Left	Center	Right
Top	1	2	-1
Middle	-5	0	20
Bottom	1	1	1

You are Row. Consider playing the first strategy (top). If your opponent could read your mind, then she would play her third strategy (right). She would win 1 and you would lose 1. What about playing your second strategy (middle)? If your opponent knew, then your payoff would be -5 since she would pick her first strategy (left) as the response. Finally, the third strategy (bottom) pays you 1 no matter what your opponent does. Therefore, if you are conservative (or paranoid) or really playing against an omniscient opponent, then you would play the third strategy. The third strategy establishes your **pure-strategy security level**. Informally, the pure-strategy security level is the amount that you can guarantee for yourself no matter what your opponent does. The reason for the modifier "pure strategy" will become clear soon. Formally, your security level is $\max_i \min_j u(i, j)$.

Take a moment to analyze this expression. Define an intermediate function: $f(i) = \min_j u(i, j)$. $f(i)$ is what you would get if you played your ith strategy and your opponent made the response that was the worst for you (and the best for her). Your security level is $\max_i f(i)$. That is, it is the maximum payoff you get assuming that your opponent will observe your strategy choice and take full advantage of this information.

A security level gives a lower bound to your payoff in the game. Surely you should expect to do no worse than this when you play the game. Can you expect to do better?

One way to check is to put yourself in the position of the column player. She too can try to guarantee her security level. In the example, when she plays left, the worst that can happen is that she gets -1 (if Row plays top or bottom); when she plays center, the worst that can happen is that she gets -2; when she plays her right, the worst that can happen is that she gets -20. (Remember that the payoff that Column gets is the negative of the payoff that Row gets.) So, Column's pure-strategy security level is -1, which she'll get if she plays her first strategy. In this game, at least, it appears that the security level is a good prediction of the value of the game. Row player can play in such a way that guarantees that he will win 1. Column player can play in such a way that guarantees that she will lose no more than -1. Since it is a zero-sum game, everything that Row wins must come from Column. Hence if Column plays to guarantee her security level, then Row cannot win more than 1. If Row plays to guarantee her security level, then Column must lose at least 1. There is no room for either player to do better than their security level.

If the result of the example were general, then we would have a good theory of how to play zero-sum games. Row player should play to insure that he obtains his security level because if his opponent plays sensibly the Row can do no better than obtaining his security level. The same statement holds for the Column player. So, for the second time, is this general? The answer is yes and no.

First, here is the reason why the answer is no. Take matching pennies. The pure strategy security level for both players is -1 (and the players can attain this payoff by using either strategy). I hope that the reason for this is easy to understand. If your opponent could figure out how you were going to play this game, then she would always win. So it is too conservative to play as if your opponent can out guess you.

Now imagine you were going to play the matching pennies repeatedly with the same person. If you were to play the same strategy each time you played the game, what would you do? You probably would not want to be predictable. That is, you probably would not want to play heads every time. If you did, then there is a chance that your opponent could figure that out and take advantage of you. The notion of a mixed strategy, is a way to describe the idea of being unpredictable. For example, suppose that instead of deciding whether to play heads or tails, you simply flip the coin and play whatever side lands face up. In this way, you end up playing heads half of the time and tails half of the time (I am assuming that your penny is a fair coin that lands heads half the time). You would like to know what your payoff would be if you followed this strategy. In order to figure this out, you need to know two things. First, you need to understand that you must be content to compute your expected payoff (if half of the time you play heads and the other half you play tails, then you won't always win or always lose). Second, you need to make some assumption about your opponent's play in order to figure out your payoff.

You answer the first question by computing expected payoffs. Doing so requires that you interpret the numbers in the payoff matrix as utilities and that these utilities satisfy the expected utility property. You either learned all about this in Econ 171, will learn all about this in Econ 171, or will live an empty, unhappy existence. Here I will say that there is a well-developed theory of decision making under uncertainty that gives conditions under which using expected utilities is justified. This theory is a bit controversial, but is still the standard way of treating payoffs in games.

Once you interpret the numbers in the payoff matrix as expected utility, you must remind yourself that they need not be monetary payoffs. A player need not be indifferent between winning nothing or a $50 - 50$ gamble than pays 1 when it wins and costs 1 when it loses. In fact, someone who is risk averse, strictly prefers to avoid the gamble. However, a player must be indifferent between a gamble that either gives zero utility or a $50 - 50$ gamble that pays utility of 1 or utility of -1.

The second issue is to decide how to evaluate the payoff associated with playing the random (or mixed) strategy of playing head and tails with equal probability. The answer is to do what we did with pure strategies. Suppose Column knows that Row is going to play heads and tails with equal probability. What is the worst think that she can do (from Row's point of view)? The answer is that it does not matter what Column does. Row's expected payoff is always zero. Of course, a symmetric argument establishes that by playing heads and tails with equal probability that Column could also guarantee herself a payoff of zero.

The example illustrates the idea of the **mixed-strategy security level**. A **mixed strategy** is a probability distribution over pure strategies. In games with two pure strategies, like matching pennies, a probability distribution can be described by a number p between zero and one (interpret p as the probability that the player plays his first strategy, so that $1 - p$ is the probability that he plays his remaining strategy). In general, if the player has n pure strategies, the mixed strategy is a vector $p = (p_1, \ldots, p_n)$ such that $p \geq 0$ and $\sum_{i=1}^{n} p_i = 1$, where you interpret p_i as the probability that the player picks his ith pure strategy. The mixed strategy security level of the Row player is defined as

$$\max_p \min_q \sum_{i=1}^{n} \sum_{j=1}^{m} p_i q_j u(i,j).$$

In this expression, I assume that Row has n pure strategies, Column has m pure strategies, and that p and q are mixed strategies for Row and Column respectively. It is convenient to let U be a matrix with n rows and m columns (typical entry u_{ij}). In that case the security level is

$$\max_p \min_q pUq$$

The mixed-strategy security level of the Column player has a similar definition:

$$\max_q \min_p \sum_{j=1}^{m} \sum_{i=1}^{n} -p_i q_j u(i,j).$$

We can write this as

$$\max_q \min_p -pUq = -\min_q \max_p pUq.$$

This definition reverses the order of p and q and puts a minus sign in front of $u(i,j)$ (because the payoff of Column is equal to -1 times the payoff of Row).

You should be able to convince yourself that the mixed-strategy security level is at least as great as the pure-strategy security level. (In matching pennies, Row's pure-strategy security level is -1 while his mixed-strategy security level is zero.) The intuition for this is that in figuring out what to do, the Row player has the choice of using a "degenerate" mixed strategy that places probability one on a pure strategy. Having the extra option of randomizing couldn't make him worse off.

Warning: From now on, when I say security level I will mean mixed-strategy security level.

Only slightly less obvious is the assertion that if you add Row's security level to Column's security level you get something that is less than or equal to zero. In symbols:

$$\max_p \min_q pUq - \min_q \max_p pUq \leq 0.$$

This inequality merely expresses the idea that it is possible for both Row and Column to attain their security levels (since the payoff sums must be equal to zero, if the sum of the security levels were negative it would be impossible for both players to get their security level). U is the payoff matrix.

The fundamental theorem of two-player zero-sum games is that the inequality above must actually hold as an equation. In symbols, the fact is that

$$\max_p \min_q pUq = \min_q \max_p pUq.$$

This fact is called the Minimax Theorem. In words it says that if Row plays in such a way that guarantees his security level, then Column cannot get more than her security level. Also, if Column plays in such a way that guarantees her security level, then Row cannot get more than his security level. Hence the Minimax Theorem tells you how you should play zero-sum games (at least against a "sensible" opponent): Each player should play to maximize his or her security level. Why? One answer is that it maximizes your minimum expected payoff. That is, there is a sense in which it is safe. This answer is not compelling on its own. It becomes compelling in zero-sum games because the Minimax Theorem says that you can only expect more than your security level if your opponent gets less than her security level. One should not expect a sensible player to settle for less than what she could guarantee for herself. If you do assume that your opponent is sensible in this way, then you cannot hope to do better than your security level. Hence playing a strategy that guarantees your security level is the right way to play the game.

Remember: Mixed-strategy security levels are expected utilities. In matching pennies you never get a payoff of exactly zero. Each time you play the game you either win or you lose. However, if you play heads and tails with equal probability, then your expected payoff is zero. Once again, it is essential to remember that the payoffs are utilities. Although you are unhappy when you lose, before you play you are indifferent between actually playing matching pennies or not playing at all.

The recommendation that you play a strategy that guarantees your security level is appropriate if your opponent is sensible. If your opponent does not play a sensible strategy, then it might be appropriate to play something besides your minimax strategy to exploit his stupidity. Specifically, if your opponent (Row) always plays heads in matching pennies, then it would be silly for you to randomize. You should play tails and win for sure.

6 Linear Programming and Zero-Sum Game Theory

I haven't forgotten that this is a course in Linear Programming. You figured that there was just some extra time to kill and so I threw in an unrelated topic. But no.

The Minimax Theorem is a simple consequence of the Duality Theorem of Linear Programming. Seeing the relationship allows you to use Linear Programming techniques to solve zero-sum games.

You should not be surprised to find that there is a relationship. Aside from the cynical reasons (you figured that I wouldn't stray too far from the main topic of the course), there is obviously something linear going on in the problems that define security level. Furthermore, the maxmin objective (pUq looks a lot like the yAx object that appeared in our discussions of duality and complementary slackness).

Consider the following pair of LPs:

$$\max w \text{ subject to } pU - we \geq 0, p \cdot e = 1, p \geq 0.$$

$$\min v \text{ subject to } Uq - ve \leq 0, q \cdot e = 1, q \geq 0.$$

In these problems, e is a vector of ones (be careful, sometimes e has n ones, sometimes m ones, depending on context. To test your understanding, figure out which is which). In the first problem, the variables are w (a real number) and p (an n-dimensional vector). The first constraint says that each component of pU (there are m of them) should be greater than or equal to w. The second and third constraints state that p should be a probability distribution for the Row player (a mixed strategy). Suppose that the Row player uses the mixed strategy p. If the column player could observe this choice, then she could compute her payoff for any strategy. pU is an m vector, the jth component of which gives the expected payoff to Row if Row plays p and Column picks her jth pure strategy (Column gets -1 times this). Hence if $pU - we \geq 0$, then Row gets at least w (no matter what Column does) when he uses p. It follows that the solution (p^*, w^*) to the first LP above gives Row's security level (w^*) and a strategy that attains the security level (p^*). Similarly, the second problem gives Column's security level. Careful: The second problem does describe how to find Column's security level, but the value of the problem actually gives the payoff to the Row player. That is, if (q^*, v^*) is the solution to the second problem, then the security level of the column player is $-v^*$.

Verifying that the problems are dual and confirming that the relationship they have to security level is a bit confusing. It is a worthwhile exercise to verify the relationship carefully.

A bit of careful accounting (no thinking, just remembering the definition of dual linear programming problems) confirms that the second problem is the dual of the first problem. Since both problems are feasible (for example, in the first problem let p be anything that satisfies the second and third constraints and let w be the smallest element in U), the Duality Theorem states that both problems have solutions. and the values are the same.

7 Examples Revisited

7.1 Colonel Blotto

The first thing to notice about Colonel Blotto is that the Colonel has two dominated strategies: It is never in his interest to allocate all of his troops to one mountain pass. He can successfully defend the pass with only two troops. This intuitive conclusion also follows from an examination of the payoff matrix. The payoffs in the first row are all lower (or equal) to the numbers in the same column in the second row. Hence we can reduce the game to:

	(2,0)	(1,1)	(0,2)
(2,1)	1	1	-1
(1,2)	-1	1	1

If you study this game, you will see that the enemy's middle strategy $(1, 1)$ is now dominated. The enemy will never win by sending only 1 division to a location because Colonel Blotto has at least one division at both passes. Deleting this strategy simplifies the game even further:

	(2,0)	(0,2)
(2,1)	1	-1
(1,2)	-1	1

This game looks like matching pennies, so we know that the best strategy is for both players to randomize $50 - 50$ over their (remaining) strategies.

It should not surprise you that there are many variations of the game (depending on the number of divisions the two sides have; what it takes to win a battle; and what it takes to win the war).

7.2 Morra

To compute the pure-strategy security level in Morra, note that if your opponent knew you were playing 12, she'd play 23; if she knew that you were playing 13, then she'd play 12; if she knew you were playing 23, then she'd play 24; and if she knew that you were playing 24, then she'd play 13. In each case, she'd win. Her winnings would be at least \$2 (if you played 13), so your pure-strategy security level is $-\$2$. The game is symmetric, so the column player can guarantee that she loses no more than \$2 as well, but cannot do better. Hence there is a gap between the pure-strategy security levels. The equilibrium strategy must be mixed. This conclusion is not surprising. Mixed-strategies appear in situations where you do not want your opponent to be able to predict your behavior. There are infinitely many mixed strategies that lead to an expected payoff of at least zero. One possibility is to play 13 with probability .6 and 23 with probability .4 (and the other strategies with probability 0). If the other player plays 12, then you lose 2 with probability .6 and you win three with probability .4. You're expected payoff is zero. If the other player plays 13, then you always break even (you both guess right when you play 23 and you both guess wrong when you play 13). Similarly, if your opponent plays 23, then you always break even. Finally, if your opponent plays 24, you have an expected gain of $\$3(.6) - \$4(.4) = \$.20$. It follows that if you play the indicated mixture, then you guarantee a non-negative expected payoff. Your expected payoff will be positive if your opponent plays 24. It is not surprising that your expected payoff is zero. It is not surprising that you play randomly. It may be surprising to learn that you should avoid using the strategy 24 (even though this is the only strategy that gives you a chance of winning \$4). Of course, if you think that your opponent is likely to play 23 with high probability, then you should not reject 24. The analysis implicitly demonstrates that it is not prudent to play 23 with high probability, however.

7.3 Goofspiel

I do not have the energy to write down a 24×24 payoff matrix for goofspiel with $n = 3$. We can confirm that a particular strategy is optimal for the $n = 3$ game assuming $a_i = i$ for $i = 1, 2, 3$. I claim that in this game an equilibrium strategy is to play card i at move i. Suppose you play this strategy. If your opponent plays 3 on the last move, then you are guaranteed a payoff of 0 (you break even in the last round and you either tie in the first two rounds or lose the first and win the more valuable second round). If your opponent does not play 3 on the final round, then you win 3 on the final round and even if you lose the first two rounds you still break even for the entire game. Hence your security level is at least zero. Your opponent can play in the same way, however. Therefore, her security level is also zero. It must be that the value of the game is zero and (against a rational opponent) you can do no better than to play the strategy that I described. The strategy of playing card i at round i would continue to be optimal if you increased a_3 while keeping a_1 and a_2 constant. On the other hand, if you decreased the relative importance of the third round, say by having $a_i = i + 3$ for $i = 1, 2, 3$, then it would not be rational to always play 3 on the third move.

8 More Examples

1. Calculate the optimal strategy.

	1	2	3
1	16	-8	4
2	-24	-16	3
3	1	1	2

Here by playing the third strategy, Row can guarantee a payoff of one. On the other hand, column can hold player one to this payoff by playing Column 2. Hence the game has a pure strategy equilibrium point. The value of the game (to the Row Player) is 1.

2. Each of two players has a Ace, King, Queen, and Jack. They each simultaneously show a card. Player 1 wins if they both show an Ace, or if neither shows an Ace and the cards do not match. Player 2 wins if exactly one shows an Ace or if neither shows an Ace and the cards match. (The winner receives a payment of $1 from the loser.)

   I will treat Player 1 as the row player and Player 2 as the column player. The payoff matrix looks like this:

	ACE	KING	QUEEN	JACK
ACE	1	-1	-1	-1
KING	-1	-1	1	1
QUEEN	-1	1	-1	1
JACK	-1	1	1	-1

Using pure strategies, neither player can guarantee a win. The pure strategy security level for each player is therefore -1. Figuring out the mixed strategy security level "by hand" for a game with four strategies for each player is tedious and hard. In this example, three of the strategies are symmetric. This suggests a simplification.

Assume that Player 1 always plays Jack, Queen, and King with the same probability (this probability can be any number between 0 and $\frac{1}{3}$; if the probability is 0, then Player 1 always plays ACE; if the probability is $\frac{1}{3}$, then Player 1 never plays ACE, and plays each of the remaining cards with the same probability. You can view the game as having two pure strategies for Player 1 (either he plays ACE or he doesn't) and the payoff matrix becomes:

	ACE	KING	QUEEN	JACK
ACE	1	-1	-1	-1
NOT ACE	-1	$\frac{1}{3}$	$\frac{1}{3}$	$\frac{1}{3}$

.

If you impose the same symmetry condition on Player 2, the game reduces to:

	ACE	NOT ACE
ACE	1	-1
NOT ACE	-1	$\frac{1}{3}$

.

It is now possible to find the mixed-strategy equilibrium for this 2×2 (that is, two strategies for each player) game. Player 1's strategy will equalize the payoff he gets from either strategy choice of Player 2. That is, the probability of ACE, call it a will satisfy:

$$a + (1 - a)(-1) = a(-1) + (1 - a)\frac{1}{3}.$$

The solution to this equation is $a = \frac{2}{5}$. When Player 1 mixes between ACE and NOT ACE with probabilities $\frac{2}{5}$ and $\frac{3}{5}$ he gets a payoff of $-\frac{1}{5}$ whatever Player 2 does. Similarly, you can solve for Player 2's strategy by equalizing Player 1's payoff. If b is the probability that Player 2 plays ACE, then b should satisfy:

$$b + (1 - b)(-1) = b(-1) + (1 - b)\frac{1}{3}$$

or $b = \frac{2}{5}$.

Now you can go back and check that the symmetry assumption that I imposed is really appropriate. Both players are playing ACE with probability $\frac{2}{5}$ and the other strategies with total probability $\frac{3}{5}$. That means that they play each of the non-ACE cards with probability $\frac{1}{3}$. Under this condition Player 1 expects to earn $-\frac{1}{5}$ from each of his original pure strategies, and Player 2 can hold Player 1 to this amount by playing ACE with probability $\frac{2}{5}$ and the other three cards with probability $\frac{1}{5}$ each.

3. Player I's payoff matrix in a zero-sum game is:

TOP	1	2	3	4	5
BOTTOM	9	7	5	3	1

Find the pure and mixed strategy security levels of each player and the equilibrium.

Using pure strategies, player I can guarantee a payoff of 1 (using either strategy). Player 2 can guarantee a loss of no more than 4 (by playing the fourth column). This means that the game must have only a mixed-strategy equilibrium. You have seen formulas that determine the best mixed strategy (and you could find it using Excel), but when one of the players has only two strategies there is a graphical way of finding the solution. I will describe the process (come to lecture to see the picture). On the x axis you denote the probability that the row player plays up. This number goes from zero to one. Next graph the payoff associated with each of the column player's pure strategies. Column one will be a line segment that starts at $(0, 9)$ (the row player gets 9 if row plays up with probability zero) and goes to $(1, 1)$ (the row player gets 1 if row plays up with probability one). Do this for all of the strategies of the Column player. So you get five line segments. For each x, the worst that Row can do is the lowest of the five segments. Form a curve determined by the minimum of the segments. The highest point on this curve is Row's security level. For this example, all five segments intersect at the same point, $x_1 = \frac{2}{3}$. This is the point that leads to the highest value for Row. Hence Row's security level is $\frac{11}{3}$. There are many ways in which Column can hold Row to this level. One way is for Column to play column 3 with probability $\frac{1}{3}$ and column 4 with probability $\frac{2}{3}$. In this example, it is an accident that all payoffs are equal at the same mixed strategy.

Algebraically, you can solve the problem like this. Notice that the higher the probability that Row player UP, the more attractive it is for Column to play "left" columns. If Column was sure that Row would play UP, then Column would play Column 1. As the probability of playing up drops, the second column becomes a more attractive strategy. At some point, call it x_1 Columns 1 and 2 yield the same payoff. The defining condition is

$$1x_1 + 9(1 - x_1) = 2x_1 + 7(1 - x_1),$$

which implies that $x_1 = \frac{2}{3}$. You can check that this mixture guarantees Row a payoff of $\frac{11}{3}$. (Explicitly check Column's other strategies.) Since Row does worse with any other mixture, this must be his optimal strategy. It is an accident that all five of columns strategies work equally well against Row's optimal strategy. It is not an accident that the mixture that attains Row's security level makes Column indifferent between at least two strategies.

4. Consider the following game.

	LEFT	CENTER	RIGHT
TOP	1	2	4
BOTTOM	9	5	1

This game also has no equilibrium in pure strategies. Row's (pure-strategy) security level is 1, while Column can hold Row to 4 by playing the right column. Figure out the probability x_1 of playing UP that equalizes the payoffs of the first two columns in the table:

$$x_1 + 9(1 - x_1) = 2x_1 + 5(1 - x_1),$$

or $x_1 = \frac{4}{5}$. When Row uses this mixture, the third column is strictly better for Row (payoff $\frac{17}{5}$) than either of the first two columns. Hence Row can get at $\frac{13}{5}$ if he plays UP with probability $\frac{4}{5}$. Row would not be guaranteed to do better if he played UP with higher probability. If Column knows that Row will play UP with higher probability, then Column would play LEFT, leading to a payoff for Row of less than $\frac{13}{5}$. So we have ruled out mixed strategies with probability greater than $\frac{4}{5}$ on UP. What about other mixtures? For these, Column is likely to respond with either the Center or Right column. Figure out the probability x_2 of playing UP that equalizes the payoffs of the last two columns:

$$2x_2 + 5(1 - x_2) = 4x_2 + (1 - x_2),$$

or $x_2 = \frac{2}{3}$. If Row uses this mixture, then he is guaranteed a payoff of 3, which he will get if Column plays either his second or third strategy. (Row will do even better if Column plays the left column.) Furthermore, if Row places less weight on UP, Column will be able to reduce Row's payoff by playing RIGHT, and if Row plays up with a probability between x_1 and x_2, then Column will hold Row's payoff below 3 by playing CENTER. Since $3 > \frac{13}{5}$, Row's security level must be 3; his equilibrium strategy is to play up with probability $\frac{2}{3}$. Column can hold player one to this payoff by mixing between CENTER and RIGHT, playing each with probability $\frac{1}{2}$.