Every Choice Correspondence is Backwards-Induction Rationalizable∗

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Abstract

We extend the result from Bossert and Sprumont (2013) that every single-valued choice function is backwards-induction rationalizable via strict preferences to the case of choice correspondences via weak preferences.

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∗In a different paper, Xiong (2014) independently derived the same result
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1 Introduction

Bossert and Sprumont (2013) define a choice function on the outcomes of an extensive-form game as *backwards-induction rationalizable* “if there exists a finite perfect-information extensive-form game such that, for each subset of alternatives of the extensive-form game, the backwards-induction outcome of the restriction of the game to that subset of alternatives coincides with the choice from that subset.” Bossert and Sprumont (2013) then prove that for finite-alternative extensive-form perfect-information games, every choice function is backwards-induction rationalizable via strict preference relations. One easily finds this unique backwards-induction outcome using the backwards-induction algorithm developed in Kuhn (1953) for finite extensive-form games with strict preferences.

Some examples of choice functions studied by Bossert and Sprumont (2013) on the set of alternatives $A = \{1, 2, 3\}$ are given by $f_1$ and $f_2$ in Figure 1. These examples show that the choice functions that are backwards-induction rationalizable can be irregular as both $f_1$ and $f_2$ violate Sen’s $\alpha$ property as defined in Sen (1970). In particular, the choice function $f_2$ chooses alternative 3 from the universal set although it is never chosen from any pair of alternatives which would be quite irregular. However, Bossert and Sprumont do not allow weak preferences and do not consider choice correspondences. The purpose of this note is to establish that all choice correspondences are backwards-induction rationalizable via weak preference relations. The choice correspondences which are backwards-induction rationalizable can also be irregular as seen by the choice correspondences $f_3$ and $f_4$ in Figure 1. The correspondences are irregular since $f_3$ violates Sen’s property $\alpha$ and $f_4$ violates Sen’s property $\alpha$ and $\beta$ with these properties as defined in Sen (1970).

This paper contributes to a growing literature which seeks to identify the testable aspects of joint decision making when preferences are unobserved. Some relevant literature includes Sprumont (2000) and Galambos (2005) which examine choice functions and Nash equilibria of

![Figure 1: Examples of the three-alternative case](image)

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2 Definitions

Presented here are the relevant definitions from Bossert and Sprumont (2013). First, let $A = \{1, \ldots, a\}$ be the universal set of alternatives. The power set of $A$ excluding the empty set is denoted by $\mathcal{P}(A)$. A choice correspondence $f$ on a set of alternatives $A = \{1, \ldots, a\}$ is a mapping $f: \mathcal{P}(A) \Rightarrow A$ such that for every $B \in \mathcal{P}(A)$, one has that $f(B) \subset B$ and $f(B) \neq \emptyset$ where “$\subset$” denotes weak set inclusion. Here we focus on choice correspondences which are defined for all subsets of the universal set.

A binary relation $\succeq$ on a set $X$ is complete if for all $x, y \in X$, $x \succeq y$ or $y \succeq x$. The relation $\succeq$ is transitive if for all $x, y, z \in X$, then $[x \succeq y$ and $y \succeq z]$ implies $x \succeq z$. The relation $\succeq$ is asymmetric if for all $x, y \in X$, $x \succeq y$ implies that $y \succeq x$ does not hold. Finally, $\succeq$ is antisymmetric if for all $x, y \in X$, $[x \succeq y$ and $y \succeq x]$ implies that $x = y$. Let $\succ$ be a transitive and asymmetric precedence relation on a non-empty and finite set $N$. We define that $n \in N$ is an immediate predecessor of $n' \in N$ if $n \prec n'$ and there does not exist an $n'' \in N$ such that $n \prec n'' \prec n'$. Similarly, we say that $n \in N$ is an immediate successor of $n' \in N$ if $n' \prec n$ and there does not exist an $n'' \in N$ such that $n' \prec n'' \prec n$. The set of
immediate predecessors of a node $n \in N$ is defined as $P(n)$, and $S(n)$ is defined as the set of immediate successors of the node $n$.

A tree, $\Gamma$, is given by a quadruple $(0, D, T, \prec)$ where:

1. $0$ is the root;
2. $D$ is a finite set of decision nodes such that $0 \in D$;
3. $T$ is a non-empty and finite set of terminal nodes such that $D \cap T = \emptyset$;
4. $\prec$ is a transitive and asymmetric precedence relation on $N = D \cup T$ such that:
   - $|P(0)| = 0$ and $|S(0)| \geq 1$;
   - $\forall n \in D \setminus \{0\}, |P(n)| = 1$ and $|S(n)| \geq 1$;
   - $\forall n \in T, |P(n)| = 1$ and $|S(n)| = 0$.

Note that when comparing two trees $\Gamma_a$ and $\Gamma_b$, we denote the components $\Gamma_a = (0_a, D_a, T_a, \prec_a)$ and $\Gamma_b = (0_b, D_b, T_b, \prec_b)$. Similarly for $n \in N_a$ we use the notation $P_a(n)$ and $S_a(n)$, while for $n \in N_b$ we use $P_b(n)$ and $S_b(n)$. We also say that a path in $\Gamma$ from a decision node $n \in D$ to a terminal node $n' \in T$ (of length $K \in \mathbb{N}$) is an ordered $(K+1)$-tuple $(n_0, \ldots, n_K) \in \mathbb{N}^{[K+1]}$ such that $n = n_0$, $\{n_{k-1}\} = P(n_k)$ for all $k \in \{1, \ldots, K\}$, and $n_K = n'$.

A game on the set of outcomes $A = \{1, \ldots, a\}$ is a triple $G = (\Gamma, g, R)$ where:

1. $\Gamma$ is a tree;
2. $g: T \to A$ a function such that, for all $n \in T$, $g(n) \in A$;
3. $R: D \to R_A$ is a preference assignment map which assigns a complete and transitive preference relation to each decision node, where $R_A$ is the set of orderings on $A$.

Once again, we use the notation of $(\Gamma_a, g_a, R_a)$ for a game $G_a$. Let $G$ be a game on $A$. For any $B \in \mathcal{P}(A)$ such that $B \subset g(T)$, the restriction of $G$ to $B$ is the game $G|B = G_r = (\Gamma_r, g_r, R_r)$ on the set of alternatives $B$ such that:
(1) $0_r = 0$;

(2) $D_r = \{ n \in D \mid \text{there exists } n' \in g^{-1}(B) \text{ and a path in } \Gamma \text{ from } n \text{ to } n' \}$;

(3) $T_r = g^{-1}(B)$;

(4) $\prec_r$ is the restriction of $\prec$ to $N_r = D_r \cup T_r$;

(5) $g_r$ is the restriction of $g$ to $T_r$;

(6) $R_r : D_r \to \mathcal{R}_B$ is a restricted preference assignment map of $R$, where $\mathcal{R}_B$ is the set of orderings on $B$.

An example of a game $G$ on a set $A = \{1, 2, 3, 4\}$ that has the restricted game $G' = G|B$ where $B = \{1, 3\}$ is shown in Figure 2. In this game arrows point from terminal nodes to their corresponding outcome for a given outcome function and the preference relations are omitted in the figure.

![Figure 2: A game on $A = \{1, 2, 3, 4\}$ restricted to $B = \{1, 3\}$](image)

Next, we denote $\max(B; \tilde{R})$ to be the set of best elements in $B \in \mathcal{P}(A)$ according to the complete and transitive preference relation $\tilde{R} \in \mathcal{R}_B$. For a game $G$ and each decision node $n \in D$, we denote $e_n(G)$ as the set of backwards-induction outcomes of the subgame rooted at $n$. This set of outcomes is defined in the expected way where we first set $e_n(G) = g(n)$ for all $n \in T$. Then we recursively define $e_n(G) = \max(\{e_{n'}(G) \mid n' \in S(n)\}; R(n))$ for all $n \in D$. To simplify notation we define $e(G) = e_0(G)$. Note that the set of backwards-induction outcomes for any subgame of $G$ (including $G$ itself) exists since $G$ is a finite-alternative perfect-information game. Similarly, for every $B \in \mathcal{P}(A)$, the set of backwards-induction
outcomes for $G|B$ is also well defined.

Now we extend the definition of backwards-induction rationalizability from Bossert and Sprumont (2013) to correspondences in a natural way. A choice correspondence $f$ on $A$ is \textit{backwards-induction rationalizable} if there exists a game $G$ on $A$ such that $f(B) = e(G|B)$ for all $B \in \mathcal{P}(A)$. In this case, we say that $G$ is a \textit{backwards-induction rationalization} of $f$ or that $G$ \textit{backwards-induction rationalizes} $f$.

\section{Result}

While the proof of the main result that follows is heavy in notation, the idea is very simple. First, from Bossert and Sprumont (2013) we know that for any choice functions $f$ and $g$ on some set of finite alternatives $A$ we can create extensive-form games $F$ and $G$ with strict preferences which backwards-induction rationalize $f$ and $g$ respectively. Now, we can create a new root node which exhibits complete indifference between the full set of alternatives in $A$ and then append the games $F$ and $G$ to the root node to create a new extensive-form game. This new extensive-form game includes the backwards-induction rationalizable outcomes from $f$ and $g$ and backwards-induction rationalizes the choice correspondence $f \cup g$. Similarly, we can repeat this process with any finite number of functions $\{h_i\}_{i=1}^m$ to generate larger correspondences. With this intuition in mind, we proceed to the statement and proof of the main result.

\textbf{Theorem 1.} Every choice correspondence is backwards-induction rationalizable.

\textit{Proof.} Let $A = \{1,\ldots,a\}$ be a finite set of alternatives and $\mathcal{P}(A)$ be the power set of alternatives not including the empty set. Next, let $f$ be a choice correspondence $f : \mathcal{P}(A) \supseteq A$ such that for all $B \in \mathcal{P}(A)$ we have that $f(B) \subset B$ where $f(B) \neq \emptyset$. 


Then for each set $B \in \mathcal{P}(A)$, let $f(B) = \{b_1, \ldots, b_{|f(B)|}\}$. Now, let $m = \max_{B \in \mathcal{P}(A)}\{|f(B)|\}$. Then construct the following $m$ functions $f_k$ for $k \in \{1, \ldots, m\}$ to be defined on each $B \in \mathcal{P}(A)$ as follows. For a given $B$, if $k < |f(B)|$ then let $f_k(B) = b_k$. If $k \in \{|f(B)|, \ldots, m\}$ let $f_k(B) = b_{|f(B)|}$.

We now have for each $B \in \mathcal{P}(A)$ that $\bigcup_{k=1}^m f_k(B) = \bigcup_{j=1}^{|f(B)|} \{b_k\} = f(B)$. Therefore, we have that $f = \bigcup_{k=1}^m f_k$. We claim that $f$ is backwards-induction rationalizable. From Bossert and Sprumont (2013), one has that each $f_k$ is backwards-induction rationalizable by an extensive-form game $G_k = (\Gamma_k, g_k, R_k)$ with a tree $\Gamma_k = (0_k, D_k, T_k, \prec_k)$ where $R_k$ is a preference mapping which assigns strict (antisymmetric) preference relations on nodes in $D_k$. We also extend all mappings so that $\prec_k(n) = \emptyset$ for $n \notin D_k$, $g_k(n) = \emptyset$ for $n \notin T_k$, and $R_k(n) = \emptyset$ for $n \notin D_k$. Now let $0_f$ be a new node and assign to it the universal indifference relation $R_0$ on $A$. Also, define a precedence relation $\prec_0$ which assigns $0_f$ as the unique immediate predecessor of $0_k$ for all $k \in \{1, \ldots, m\}$ and the empty set otherwise.

Then let $\Gamma_f = (0_f, D_f, T_f, \prec_f)$ where we define that $D_f = \{0_f\} \cup (\bigcup_{k=1}^m D_k)$, $T_f = \bigcup_{k=1}^m T_k$, and $\prec_f = \bigcup_{k=0}^m \prec_j$. Lastly, define the extensive-form game associated with the tree $\Gamma_f$ by $G_f = (\Gamma_f, g_f, R_f)$ where $g_f = \bigcup_{k=1}^m g_k$ and $R_f = \bigcup_{k=0}^m R_j$.

Now notice for the extensive-form game $G_f$, any set of alternatives $B \in \mathcal{P}(A)$, and for all $k \in \{1, \ldots, m\}$ that $e_{0_k}(G_f|B) = b_k$ since the subgame rooted at $0_k$ is just a finite-alternative extensive-form game with strict preferences. Also, since the preference relation $R_0$ at the root node $0_f$ exhibits complete indifference we have that $e_{0_k}(G_f|B) = b_k \in e(G_f|B)$ for all $k \in \{1, \ldots, m\}$. Therefore, we have that $e(G_f|B) = \bigcup_{k=1}^m e_{0_k}(G_f|B) = \bigcup_{k=1}^{|f(B)|} \{b_k\} = f(B)$ from our construction. Since this holds for all $B \in \mathcal{P}(A)$ the game $G_f$ is a backwards-induction rationalization of the choice correspondence $f = \bigcup_{k=1}^m f_k$. Therefore, $f$ is backwards-induction rationalizable. \qed
4 Conclusion

In this paper, we have shown that the result from Bossert and Sprumont (2013) that every choice function is backwards-induction rationalizable via strict preferences can easily be extended to choice correspondences with weak preferences by a simple constructive procedure. From the construction, we also note that this result immediately applies to the case when the choice correspondence takes a subset of choice sets as the domain. It is interesting to note that the games which backwards-induction rationalize a choice correspondence only require a single node exhibiting indifference regardless of the choice correspondence.

References


