1.) (20 points) Consider the following equations:
\[
y = X\beta + \varepsilon
\]
\[
b = (X'X)^{-1}X'y.
\]
Here \(y\) is a \((T \times 1)\) vector of observations on the dependent variable, \(X\) is a \((T \times k)\) matrix of observations on the explanatory variables with rank \(k\), and \(\beta\) is the true value of a \((k \times 1)\) vector of parameters. Let \(b_2\) denote the second element of the vector \(b\) and let \(\beta_2\) denote the second element of the vector \(\beta\). Consider the following theorem:

Let \(\hat{\beta}_2\) be any unbiased estimator of \(\beta_2\) that is constructed from some function of \(X\) and \(y\). If condition \(A\) holds, then \(\text{Var}(\hat{\beta}_2|X) \geq \text{Var}(b_2|X)\).

a.) Write down an equation for what it means to say that \(\hat{\beta}_2\) is an unbiased estimator of \(\beta_2\).

b.) State a sufficient condition or conditions \(A\) for which the above theorem is true.

2.) (30 points) Suppose that \(y_t \sim \text{i.i.d. } N(\mu, \sigma^2)\) with \(\mu \neq 0\) and let \(q_T = T/\sum_{t=1}^{T} y_t\).
Prove that \(\sqrt{T}(q_T - \mu^{-1}) \xrightarrow{L} N(0, v)\) and calculate \(v\). Does \(v = \lim_{T \to \infty} E(\sqrt{T}(q_T - \mu^{-1})^2)\)?
3.) (100 points) Suppose \( y_t = x'_t \beta + \varepsilon_t \) where \( x_t \) is a \((k \times 1)\) vector of explanatory variables and
\[
E[\varepsilon_t x_t | \varepsilon_{t-1} x_{t-1}, \varepsilon_{t-2} x_{t-2}, \ldots, \varepsilon_1 x_1] = 0
\]
with \( \{\varepsilon_t, x_t\} \) stationary and ergodic and \( E(\varepsilon_t^2 x_t' x_t') = \sigma^2 Q \). You can further assume that \( T^{-1} \sum_{t=1}^{T} x_t x_t' \overset{p}{\to} Q \) which has rank \( k \). The OLS estimates are given by \( \hat{b} = \left( \sum_{t=1}^{T} x_t x_t' \right)^{-1} \left( \sum_{t=1}^{T} x_t y_t \right) \) and \( s^2 = (T - k)^{-1} \sum_{t=1}^{T} (y_t - x'_t \hat{b}) \).

a.) Derive the asymptotic distribution of \( \hat{b} \) under the stated assumptions.

b.) Write down the formula for the standard OLS F test for testing the null hypothesis that \( \beta_1 = \beta_2 \), where \( \beta_1 \) denotes the first element of the vector \( \beta \) and \( \beta_2 \) denotes the second element.

c.) Derive the asymptotic distribution of the test you proposed in part (b) under the stated assumptions. Be sure your derivation accurately describes this particular example rather than repeating an argument or notation you may have seen in some more general case. [HINT: You can assume that \( s^2 \overset{p}{\to} \sigma^2 \) and need not prove this separately].

d.) Propose a t test you could use to test the null hypothesis that \( \beta_1 = \beta_2 \).

e.) A colleague followed the advice you gave in parts (b) and (d) but got different answers, rejecting the null hypothesis when she did test (b) but failing to reject the null hypothesis when she did test (d). What would you tell her is the most likely explanation for why she got different answers?

f.) How would you modify your answer to (b) if you instead assume that \( E(\varepsilon_t^2 x_t' x_t') = S \neq \sigma^2 Q \) where \( S \) can be consistently estimated from \( \hat{S} = T^{-1} \sum_{t=1}^{T} e_t^2 x_t x_t' \) for \( e_t = y_t - x'_t \hat{b} \)? [Note you do not have to derive the distribution, just state the form of the test statistic you would use under the new assumption]