Answer key for the midterm in 2008

1.) a.)

- There is no difference in estimating $\beta$ between Option 1 and Option 2.
- Validity of $t$ test:
  - Advantage: $t$ test using Option 2 is asymptotically valid in the presence of an unknown form of heteroskedasticity.
  - Disadvantage: If there is no heteroskedasticity, $t$ test using Option 1 may perform better than Option 2 in a small sample.

b.)

- Estimate of $\beta$:
  - Advantage: FGLS is asymptotically more efficient.
  - Disadvantage: If $V(X)$ is misspecified or not well estimated, FGLS is still consistent but may not be more efficient than OLS.
- Validity of $t$ test:
  - Advantage: If $V(X)$ is well estimated, $t$ statistic using Option 3 can be close to $t$ distribution in a small sample.
  - Disadvantage: If $V(X)$ is misspecified, $t$ test based on Option 3 is not valid.

2.) See the textbook.

3.)

a.) No. A Gaussian regression model satisfies $\epsilon|X \sim N(0, \sigma^2_I_T)$, which implies that $\epsilon$ and $X$ are independent. But

$$E\epsilon|X = E(u_t - \beta v_t)(z_t + v_t) = \sigma^2_v.$$ 

b.)

$$b = \frac{\sum_{t=1}^{T} x_t y_t}{\sum_{t=1}^{T} x_t^2} = \frac{\sum_{t=1}^{T} (z_t + v_t)(u_t - \beta v_t)}{\sum_{t=1}^{T} (z_t + v_t)^2} = \beta + \frac{T^{-1} \sum_{t=1}^{T} (z_t u_t + v_t u_t - \beta z_t v_t - \beta v_t^2)}{T^{-1} \sum_{t=1}^{T} (z_t^2 + v_t^2 + 2z_t v_t)}$$

Therefore,

$$\lim_{T \to \infty} b = \beta - \frac{\beta \sigma^2_v}{\sigma^2_z + \sigma^2_v} = \beta^*$$

c.) When a researcher is interested in forecasting $y$ with $x$.

d.) Write

$$y_t = \beta z_t + u_t$$
$$= \beta^*(z_t + v_t) + (\beta - \beta^*)z_t - \beta^* v_t + u_t$$
$$= \beta^* x_t + w_t$$

for $w_t = (\beta - \beta^*)z_t - \beta^* v_t + u_t$. Note
Since \( \mathbf{x}_t \) is a linear function of a zero-mean Normal vector, \( \mathbf{x}_t, \mathbf{w}_t \) is zero-mean bivariate Normal with covariance and variance terms as follows:

\[
\begin{bmatrix}
    x_t \\
    w_t
\end{bmatrix} = \begin{bmatrix}
    1 & 0 & 1 \\
    \beta - \beta^* & 1 & -\beta^*
\end{bmatrix} \begin{bmatrix}
    z_t \\
    u_t \\
    v_t
\end{bmatrix}.
\]

Since \( \mathbf{x}_t, \mathbf{w}_t \) is a linear function of a zero-mean Normal vector, \( \mathbf{x}_t, \mathbf{w}_t \) is zero-mean bivariate Normal with covariance and variance terms as follows:

\[
E(\mathbf{x}_t, \mathbf{w}_t) = E(z_t + v_t)[(\beta - \beta^*)z_t - \beta^*v_t + u_t]
\]

\[
= (\beta - \beta^*)\sigma_z^2 - \beta^*\sigma_v^2
\]

\[
= \left(\frac{\sigma_z^2\beta}{\sigma_z^2 + \sigma_v^2}\right)\sigma_z^2 - \left(\frac{\sigma_z^2\beta}{\sigma_z^2 + \sigma_v^2}\right)\sigma_v^2
\]

\[
= 0
\]

\[
E(w_t^2) = E[(\beta - \beta^*)z_t - \beta^*v_t + u_t]^2
\]

\[
= (\beta - \beta^*)^2\sigma_z^2 + (\beta^*)^2\sigma_v^2 + \sigma_u^2
\]

\[
= \left(\frac{\sigma_z^2\beta}{\sigma_z^2 + \sigma_v^2}\right)^2\sigma_z^2 + \left(\frac{\sigma_z^2\beta}{\sigma_z^2 + \sigma_v^2}\right)^2\sigma_v^2 + \sigma_u^2
\]

\[
= \frac{\sigma_z^2\sigma_v^2\beta^2}{\sigma_z^2 + \sigma_v^2} + \sigma_u^2.
\]

Since \( \mathbf{x}_t, \mathbf{w}_t \) is bivariate Normal with zero correlation, it follows that \( x_t \) and \( w_t \) are independent, so that

\[
E(\mathbf{x}_t \mathbf{w}_t)^2 = E(x_t^2)E(w_t^2).
\]

Since \( y_t = x_t\beta^* + w_t \), we can write

\[
b = \beta^* + \left(\sum_{t=1}^{T}x_t^2\right)^{-1}\left(\sum_{t=1}^{T}x_tw_t\right)
\]

\[
\sqrt{T}(b_T - \beta^*) = \left(T^{-1}\sum_{t=1}^{T}x_t^2\right)^{-1}\left(T^{-1/2}\sum_{t=1}^{T}x_tw_t\right).
\]

But the above calculations established that \( \{x_t, w_t\} \) is an i.i.d. variable with mean zero and variance \( Q = E(x_t^2)E(w_t^2) \), so from the CLT

\[
T^{-1/2}\sum_{t=1}^{T}x_tw_t \xrightarrow{L} N(0, Q).
\]

Also by LLN \( T^{-1}\sum_{t=1}^{T}x_t^2 \xrightarrow{p} E(x_t^2) \). Thus

\[
\sqrt{T}(b_T - \beta^*) \xrightarrow{L} N(0, A)
\]

for

\[
A = \frac{E(w_t^2)}{E(x_t^2)} = \frac{1}{\sigma_z^2 + \sigma_v^2} \left(\frac{\sigma_z^2\sigma_v^2\beta^2}{\sigma_z^2 + \sigma_v^2} + \sigma_u^2\right)
\]

e.)

\[
\sqrt{T}(b - \beta) \xrightarrow{L} N(0, \sigma_u^2/\sigma_z^2).
\]