On the Interpretation of Cointegration in the Linear-Quadratic Inventory Model

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ABSTRACT

The traditional formulation of the linear-quadratic inventory model with unit roots predicts cointegration between inventories and sales. That formulation implies that marginal production costs and the marginal benefits of inventories are both going to infinity, and the cointegrating coefficient reflects the optimal trade-off between these competing factors. This paper suggests a reformulation of the problem in which marginal production costs and marginal inventory benefits are both stationary and in which the cointegrating coefficient is the same as the value that characterizes the target inventory level in the cost function.

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1 Introduction

The linear-quadratic model of optimal inventory accumulation developed by Holt, Modigliani, Muth, and Simon (1960) and Hansen and Sargent (1980, 1981) has become the standard model for empirical analysis of inventory dynamics. Examples include Blanchard (1983), West (1986), Eichenbaum (1989), Ramey (1991), Krane and Braun (1991), Kashyap and Wilcox (1993), Durlauf and Maccini (1995), Fuhrer, Moore, and Schuh (1995), West and Wilcox (1994, 1996), and Humphreys, Maccini, and Schuh (2000). Excellent surveys of the method have been provided by West (1995), Anderson, Hansen, McGrattan, and Sargent (1996), and Ramey and West (1999).

The theoretical framework is usually developed for zero-mean, stationary variables, though the data to which the model is applied are almost always nonstationary. One popular approach is to assume that the driving variables are characterized by deterministic trends. It is then a perfectly valid procedure to apply the stationary theoretical model to the detrended data, though some efficiency can be lost by ignoring restrictions that the model may impose on the deterministic terms (West, 1989).

More care is required in accounting for stochastic trends or unit roots. The current dominant approach, developed by Kashyap and Wilcox (1993), is to assume that the firm's sales have a unit root (follow an I(1) process) whereas the unobserved shock to the firm's productivity is stationary (or I(0)). Under these assumptions, Kashyap and Wilcox show that the standard model implies that inventories and sales will be cointegrated. Further discussion and evidence on this point can be found in Granger and Lee (1989), West (1995), and Ramey and West (1999).

This note examines an unappealing feature of this cointegration that appears not to have been commented upon previously in this literature. The firm in this environment is seeing its marginal production costs go to infinity, and deliberately chooses to let its inventory management costs go to infinity so as to minimize total costs. The cointegration coefficient relating inventories to sales is determined by the optimal balancing of these two costs and the optimal rate of divergence of inventory management costs from their minimal value.

This note further suggests a simple cure for this problem. Rather than assume that unobserved productivity shocks are I(0), we propose modeling them as I(1) but cointegrated with the firm's sales. It is perhaps surprising that this alternative assumption about unobserved shocks ends up implying the identical reduced-form dynamic behavior of sales and inventories and the identical value for the sample likelihood function as in the Kashyap-Wilcox framework. However, the assumptions imply a different mapping from the reduced-form dynamic behavior into the structural parameters which would seem much more appealing theoretically. Specifically, in the framework suggested below, marginal production costs and marginal benefits of inventories are stationary along the long-run growth path, and the cointegrating relation is one and the same as the target inventory relation. The framework is suggested as a better way to interpret cointegration between inventories and sales using a coherent theoretical model of the firm's decision problem.

It is helpful first to develop this point with a simple example in which a representation of the cointegrated system and the mapping from reduced-form to structural parameters can be examined in closed form. We then show that the same results hold in general.

2 The linear-quadratic inventory model

Consider the following decision problem, similar to that in Ramey and West (1999)[1] :

$$\max_{\{Q_t,H_t\}_{t=0}^{\infty}} E_0\left\{\sum_{t=0}^{\infty}\beta^t(P_tS_t - C_t)\right\} \tag{1}$$

subject to:

$$C_t = (1/2)\left\{a_0(\Delta Q_t)^2 + a_1\left[(Q_t - U_{ct})^2 + a_2(H_{t-1} - a_4 - a_3S_t)^2\right]\right\} \tag{2}$$

$$Q_t = S_t + H_t - H_{t-1} \tag{3}$$

where

P_t = price of good

S_t = unit sales

C_t = cost of production

Q_t = quantity produced

H_t = inventories

U_{ct} = shock to marginal cost of production

[1] This is slightly different notation from that used by Ramey and West. Here production costs are given by $(1/2)a_1Q_t^2 - a_1Q_tU_{ct} + (1/2)a_1U_{ct}^2$ whereas Ramey and West specify $(1/2)a_1Q_t^2 + Q_tU_{ct}^*$. The term $(1/2)U_{ct}^2$ is from the point of view of the firm a constant which has no effect on any first-order conditions, and $-a_1U_{ct} = U_{ct}^*$ is a renormalization of the shock to marginal production costs that simplifies the algebra here. Also we specify the coefficient on inventory costs as $(1/2)a_1a_2$ whereas Ramey and West use $(1/2)a_2$. Again this is only a renormalization that simplifies the algebra, though in empirical work one might want to allow a zero coefficient on production costs and nonzero coefficient on inventory costs, which our normalization does not allow.

$$\beta = \text{discount rate.}$$

The first-order condition for cost minimization is

$$E_t[a_0(\Delta Q_t - 2\beta\Delta Q_{t+1} + \beta^2\Delta Q_{t+2}) + a_1(Q_t - U_{ct}) \qquad (4)$$
$$-\beta a_1(Q_{t+1} - U_{c,t+1}) + \beta a_1 a_2(H_t - a_4 - a_3 S_{t+1})] = 0.$$

Cost minimization must hold regardless of the relation between prices and sales. For purposes of this note, we focus exclusively on the cost-minimization condition and treat prices and sales as exogenous. The traditional interpretation of cointegration between inventories and sales arising from such a model (e.g., Kashyap and Wilcox, 1993; West, 1995, Ramey and West, 1999) comes from assuming that S_t has a unit root while U_{ct} is stationary. We begin by considering a special case for which the solution can be examined in closed form.

2.1 An example of the traditional cointegrated representation

Suppose that there are no costs of adjusting production ($a_0 = 0$) and sales follow a random walk with drift,

$$S_t = S_{t-1} + a_5 + v_{st}, \qquad (5)$$

for v_{st} white noise and $U_{ct} = v_{ct}$ is also white noise. In this case, after dividing by a_1, equation (4) becomes

$$E_t[Q_t - v_{ct} - \beta Q_{t+1} + \beta a_2(H_t - a_4 - a_3 S_t - a_3 a_5)] = 0 \qquad (6)$$

since $E_t v_{s,t+1} = E_t v_{c,t+1} = 0$. Substituting the inventory identity (3) into (6) results in

$$
\begin{aligned}
E_t[\Delta H_t + S_t - v_{ct} - \beta(\Delta H_{t+1} + S_t + a_5) & \qquad (7) \\
+\beta a_2(H_t - a_4 - a_3 S_t - a_3 a_5)] &= 0.
\end{aligned}
$$

Define

$$
\begin{aligned}
\gamma_0 &\equiv -(a_5/a_2) - a_4 - a_3 a_5 \\
\gamma_1 &\equiv \frac{1-\beta}{\beta a_2} - a_3 \qquad (8) \\
w_t &\equiv H_t + \gamma_0 + \gamma_1 S_t, \qquad (9)
\end{aligned}
$$

allowing (7) to be written

$$E_t(\Delta H_t - \beta \Delta H_{t+1} + \beta a_2 w_t - v_{ct}) = 0. \qquad (10)$$

Notice that equation (9) implies

$$\Delta H_t = \Delta w_t - \gamma_1 \Delta S_t. \qquad (11)$$

Substituting (11) into (10) results in

$$E_t(\Delta w_t - \gamma_1 \Delta S_t - \beta \Delta w_{t+1} + \beta \gamma_1 \Delta S_{t+1} + \beta a_2 w_t - v_{ct}) = 0$$

or, from (5),

$$E_t(\Delta w_t - \gamma_1 a_5 - \gamma_1 v_{st} - \beta \Delta w_{t+1} + \beta \gamma_1 a_5 + \beta a_2 w_t - v_{ct}) = 0$$

$$E_t[(1+\beta+\beta a_2) w_t - w_{t-1} - \beta w_{t+1}] = (1-\beta)\gamma_1 a_5 + \gamma_1 v_{st} + v_{ct}$$

$$E_t \beta \left[\left(1 - \frac{1+\beta+\beta a_2}{\beta} L + \beta^{-1} L^2 \right) w_{t+1} \right] = -h_0 - x_t \qquad (12)$$

for L the lag operator, $h_0 = (1-\beta)\gamma_1 a_5$ and $x_t = \gamma_1 v_{st} + v_{ct}$.

Equation (12) is an example of the well-known difference equation from Sargent (1987, p. 201). The factorization

$$(1-\lambda_1 z)(1-\lambda_2 z) = \left(1 - \frac{1+\beta+\beta a_2}{\beta} z + \beta^{-1} z^2\right) \tag{13}$$

has real roots $0 < \lambda_1 < 1$ and $\lambda_2 = 1/(\beta\lambda_1) > 1$. Recalling that x_t is white noise, the solution to (12) is known from Sargent (1987, pp. 203 and 394) to be

$$w_t = \lambda_1 w_{t-1} + \frac{\lambda_1 h_0}{1-\lambda_2^{-1}} + \lambda_1 x_t. \tag{14}$$

Notice that (14) is a stationary AR(1) process. It follows from (9) that H_t and S_t must be cointegrated with cointegrating vector $(1, \gamma_1)'$.

One can write the cointegrated system in traditional error-correction form by substituting $w_t = \Delta H_t + (H_{t-1} + \gamma_0 + \gamma_1 S_{t-1}) + \gamma_1 a_5 + \gamma_1 v_{st}$ and $w_{t-1} = H_{t-1} + \gamma_0 + \gamma_1 S_{t-1}$ into (14):

$$\Delta H_t + (H_{t-1} + \gamma_0 + \gamma_1 S_{t-1}) + \gamma_1 a_5 + \gamma_1 v_{st} = \lambda_1 (H_{t-1} + \gamma_0 + \gamma_1 S_{t-1}) + \frac{\lambda_1 h_0}{1-\lambda_2^{-1}} + \lambda_1 x_t$$

or

$$\Delta H_t = (\lambda_1 - 1)(H_{t-1} + \gamma_0 + \gamma_1 S_{t-1}) + \frac{\gamma_1 a_5 (\lambda_1 - 1)}{1-\lambda_1 \beta} + \lambda_1 v_{ct} + (\lambda_1 - 1)\gamma_1 v_{st}. \tag{15}$$

Equations (5) and (15) constitute the vector error-correction representation for the cointegrated VAR for $(S_t, H_t)'$ with cointegrating relation (9). In terms of the original parameters of the structural model defined by the optimization problem in (1)-(3), the discount rate β is unidentified on the basis of observations of $(S_t, H_t)'$ and would have to be imposed a priori. The parameter a_1 is also unidentified and would have to be normalized (say, $a_1 = 1$). The value of γ_1 is inferred from the cointegrating relation between S_t and H_t, and the value of

λ_1 is then identified from the response of inventory investment to the lagged cointegrating residual w_{t-1} or to the current shock to sales v_{st}. The framework is overidentified in that both the contemporaneous correlation between the VAR residuals in (5) and (15) and the response of ΔH_t to the lagged error-correction term w_{t-1} are governed by the single parameter λ_1. Given λ_1 and β, the value of a_2 can then be found from (13). Knowing β, a_2, and γ_1, one can then infer a_3 from (8).

A surprising feature of this solution, which has been noted by Kashyap and Wilcox (1993), West (1995), and Ramey and West (1999), is that the cointegrating vector $(1, \gamma_1)'$ is not the value $(1, -a_3)'$ that one might have expected on the basis of the underlying cost function. Indeed, from (8) and (9),

$$H_{t-1} - a_3 S_t = -\Delta H_t + w_t - \gamma_0 - \frac{1-\beta}{\beta a_2} S_t \tag{16}$$

which, since ΔH_t and w_t are I(0), must be I(1). As such, the variance of $H_{t-1} - a_3 S_t$ is $O(t)$, meaning the probability that $|H_{t-1} - a_3 S_t|$ exceeds any finite bound goes to unity as t goes to infinity. In other words, the marginal cost of managing inventories necessarily goes to infinity under the traditional formulation.

This results in general from the fact that if U_{ct} is stationary and production Q_t is I(1), then the marginal cost of production $a_1(Q_t - U_{ct})$ must go to infinity. Cost minimization calls for continually cutting back inventories relative to the target value $a_4 + a_3 S_t$ so that the infinite marginal benefit of putting another unit into inventory is equated with the infinite marginal cost of producing the good. Specifically, the marginal benefit of adding another unit to inventory (which is the negative of the marginal cost of increasing H_t) is given by

$-\beta a_1 a_2 E_t(H_t - a_4 - a_3 S_{t+1})$. From (16) this marginal benefit is asymptotically dominated by the term $\beta a_1 a_2(1-\beta)/(\beta a_2)S_t = a_1(1-\beta)S_t$. The marginal cost of producing another unit Q_t for inventory (and using the additional inventory, say, to produce one less unit next period Q_{t+1}) has a marginal cost of $E_t a_1[Q_t - U_{ct} - \beta(Q_{t+1} - U_{c,t+1})]$, which is asymptotically dominated by the term $a_1(1-\beta)S_t$. The asymptotic marginal cost equals the asymptotic marginal benefit. Hence the cointegration coefficient γ_1 results from the firm's desire to drive inventory management costs to infinity at the optimal rate given that marginal production costs are also going to infinity. Of course, this optimal plan is also causing profits to go to negative infinity, and if shutting down the plant is an option, that is obviously superior to the solution implied by the first-order conditions.

All of this would seem to be a most unappealing theoretical framework with which to account for the trends and comovement in inventories, production, and sales. The problem, moreover, is rather fundamental to the above framework. Whenever there is a unit root in sales but none in marginal production costs U_{ct}, then physical production costs must go to infinity and inventory management costs will also go to infinity under a cost-minimizing strategy. On the other hand, if one tries to fix this by assuming that U_{ct} has a unit root, if this is unrelated to the unit root in sales, then inventories and sales would no longer be cointegrated.

2.2 A more appealing cointegrated representation

The obvious solution, if one wants to account for cointegration between inventories and sales using the model (1) through (3), is to assume that U_{ct} and S_t both have unit roots, but that they are themselves cointegrated. Such an assumption might be defended on the grounds that it may be technological advance (an upward trend in U_{ct}) that generates the upward trend in sales; the appendix provides a general equilibrium example of how this would occur. Consider, for example, the consequences if we set $U_{ct} = S_t + v_{ct}$ for v_{ct} white noise but leave the other details exactly as in Section 2.1. In this case, equations (6) and (7) would become

$$E_t[(Q_t - S_t - v_{ct}) - \beta(Q_{t+1} - S_{t+1} - v_{c,t+1}) + \beta a_2(H_t - a_4 - a_3 S_t - a_3 a_5)] = 0$$

$$E_t[\Delta H_t - v_{ct} - \beta \Delta H_{t+1} + \beta a_2(H_t - a_4 - a_3 S_t - a_3 a_5)] = 0. \tag{17}$$

If we now define

$$\tilde{\gamma}_0 \equiv -a_4 - a_3 a_5$$

$$\tilde{\gamma}_1 \equiv -a_3 \tag{18}$$

$$\tilde{w}_t = H_t + \tilde{\gamma}_0 + \tilde{\gamma}_1 S_t, \tag{19}$$

expression (17) can be written

$$E_t(\Delta H_t - \beta \Delta H_{t+1} + \beta a_2 \tilde{w}_t - v_{ct}) = 0. \tag{20}$$

Equation (20) will be recognized as identical to (10). Hence the same analysis used in equations (12) through (14) can be used to establish that $\tilde{w}_t$ is stationary, meaning that $(H_t, S_t)'$ is again cointegrated, though the cointegrating vector is now $(1, \tilde{\gamma}_1)' = (1, -a_3)'$. Note that in this solution, the deviation of inventories from target and marginal production costs are both stationary.

2.3 On the consequences of using a different representation

The assumption $U_{ct} = v_{ct}$ was seen in Section 2.1 to imply the vector error-correction model

$$S_t = S_{t-1} + a_5 + v_{st} \tag{21}$$

$$\Delta H_t = (\lambda_1 - 1)(H_{t-1} + \gamma_0 + \gamma_1 S_{t-1}) + \frac{\gamma_1 a_5(\lambda_1 - 1)}{1 - \lambda_1 \beta} + \lambda_1 v_{ct} + (\lambda_1 - 1)\gamma_1 v_{st}, \tag{22}$$

whereas the assumption $U_{ct} = S_t + v_{ct}$ implies the system

$$S_t = S_{t-1} + a_5 + v_{st} \tag{23}$$

$$\Delta H_t = (\lambda_1 - 1)(H_{t-1} + \tilde{\gamma}_0 + \tilde{\gamma}_1 S_{t-1}) + \frac{\tilde{\gamma}_1 a_5(\lambda_1 - 1)}{1 - \lambda_1 \beta} + \lambda_1 v_{ct} + (\lambda_1 - 1)\tilde{\gamma}_1 v_{st}. \tag{24}$$

The observable implications of the first model for data on $\{H_t, S_t\}$ are completely described by (21) and (22), whereas the observable implications of the second model are completely described by (23) and (24). One can think of maximum likelihood estimation as first choosing parameters $(a_5, \lambda_1, \gamma_0, \gamma_1, \sigma^2_{vc}, \sigma^2_{vs})$ on the basis of the representation in (21) and (22) and then mapping these back into the structural parameters $(a_5, a_2, a_4, a_3, \sigma^2_{vc}, \sigma^2_{vs})$ as described at the end of Section 2.1. It is clear that representation (21)-(22) would imply the numerically identical value for the likelihood function as (23)-(24) and the numerically identical observed behavior for $\{H_t, S_t\}$. The key difference is the interpretation given to the estimated cointegrating coefficient γ_1. In the second model, the cointegrating relation is interpreted as directly identifying the long-run target relation between inventories and sales, $H_t = a_4 + a_3 S_t$, whereas in the first model, the cointegrating relation is interpreted as the outcome of balancing infinite inventory management costs against infinite production costs.

The two models thus predict exactly the same behavior for $\{H_t, S_t\}$, but only the first model predicts profits go to negative infinity and has the unpalatable interpretation of the long-run comovement between these two variables. It would seem clear that the second model is the better one to use for purposes of interpreting the dynamic behavior of sales and inventories, that is, one should use the structural parameter values that result from interpreting the cointegrating coefficients as $\tilde{\gamma}_0$ and $\tilde{\gamma}_1$ rather than as γ_0 and γ_1.

One objection that might be raised about the framework proposed here is that the firm is asymptotically operating around the point $Q_t = U_{ct} + k$ for some constant k implied by the other parameters. It is possible for k to be sufficiently small that the firm is predicted often to be operating in a region with decreasing marginal cost. This, however, is simple to change by replacing the assumption $U_{ct} = S_t + v_{ct}$ with $U_{ct} = S_t + a_6 + v_{ct}$. The new parameter a_6 can not be identified separately from the constant a_4 in the inventory management costs. However, if in the data $Q_t - S_t$ is stationary (as either the Kashyap-Wilcox model or the one here imply), then there exists a value of a_6 for which marginal production costs are virtually always positive, and given this value for a_6, a value for a_4 can then be found for which the model is consistent with the data. Indeed, it is precisely because our model has the feature that $Q_t - U_{ct}$ is stationary that it is possible to interpret the data as all falling in a reasonable region of the cost function. Hence, rather than a drawback of our framework, concerns about negative or infinite marginal production costs are a key reason one might prefer to base inference on the assumptions proposed here.

Note that the proposed solution only works if the cointegrating vector for $(S_t, U_{ct})'$ is

$(1,-1)'$. The general equilibrium example presented in the appendix shows why this in general would be expected to be the case.

3 A Generalization

The issues illustrated with the two examples above are in fact quite general. Consider the behavior of (4) when $(\Delta S_t, U_{ct})'$ follows an arbitrary vector I(0) process. To reproduce the argument in Kashyap and Wilcox (1993, p. 388) and West (1995, footnote 9), from the inventory identity (3), $\Delta Q_t = \Delta S_t + \Delta^2 H_t$, and if one conjectures that both ΔS_t and ΔH_t are I(0), then ΔQ_t is also I(0) and the first-order condition (4) implies that the following variable must be I(0):

$$\begin{aligned} & (Q_t - U_{ct}) - \beta(Q_{t+1} - U_{c,t+1}) + \beta a_2(H_t - a_4 - a_3 S_{t+1}) \\ = \; & (\Delta H_t + S_t - U_{ct}) - \beta(\Delta H_{t+1} + S_{t+1} - U_{c,t+1}) + \beta a_2(H_t - a_4 - a_3 S_{t+1}). \qquad (25) \end{aligned}$$

If U_{ct} is any I(0) process, then (25) can only be I(0) if

$$\beta a_2 H_t + (1 - \beta - \beta a_2 a_3) S_t \sim \text{ I}(0),$$

that is, only if $(H_t, S_t)'$ are cointegrated with cointegrating vector $(1, \gamma_1)'$ defined in (8). Hence, as noted by these authors, this vector $(1, \gamma_1)'$ is the cointegrating vector when U_{ct} is any stationary process, not just for U_{ct} white noise as discussed in the previous section.

On the other hand, if instead U_{ct} is I(1) but $S_t - U_{ct}$ is any I(0) process, then (25) is stationary provided that $\beta a_2(H_t - a_3 S_{t+1}) \sim$ I(0). Hence the result that $(1, -a_3)'$ is the cointegrating vector holds whenever S_t and U_{ct} are cointegrated with cointegrating vector

$(1,-1)'$. Thus, the approach recommended in Section 2, of replacing the assumption that U_{ct} is stationary with the assumption that $S_t - U_{ct}$ is stationary, is a general solution to the problem of interpreting cointegration between inventories and sales with a model that makes theoretical sense.

Appendix A. General Equilibrium Example

Consider a perfectly competitive representative firm with production function

$$Q_t = A_t N_t^{\alpha} \qquad 0 < \alpha < 1 \tag{26}$$

for N_t labor input (paid wage W_t) and A_t an exogenous technology shock. The firm's output is the economy's sole consumption good which is taken to be the numeraire. Suppose that the firm pays a cost X_t for inventory management,

$$X_t = \frac{\gamma}{2} \frac{(H_{t-1} - \xi S_t)^2}{S_t}, \tag{27}$$

where to obtain results parallel to those for (1)-(3), we treat this as a cost directly paid to households who supply inventory management services according to a fee schedule specified by (27).[2] The firm's decision problem is

$$\max_{\{Q_t, H_t, S_t, N_t\}_{t=0}^{\infty}} E_0 \left\{ \sum_{t=0}^{\infty} \beta^t (S_t - W_t N_t - X_t) \right\} \tag{28}$$

subject to (26), (27), and (3). Let M_t denote the marginal cost of production:

$$M_t = \frac{W_t N_t}{\alpha Q_t}. \tag{29}$$

The first-order conditions are

$$1 = M_t - \frac{\gamma}{2} \left(\frac{H_{t-1}^2}{S_t^2} - \xi^2 \right) \tag{30}$$

[2] If the firm does not make outside payments for inventory management services, then inventory management costs would need to appear in either the production function or the inventory accumulation identity. This would complicate the mapping between the general equilibrium example and the system (1)-(3).

$$M_t = \beta E_t \left[M_{t+1} - \gamma \left(\frac{H_t}{S_{t+1}} - \xi \right) \right] \tag{31}$$

$$\frac{Q_t}{S_t} = 1 + \left(\frac{H_{t-1}}{S_t} \right) \left(\frac{H_t - H_{t-1}}{H_{t-1}} \right). \tag{32}$$

To close the model, suppose that $a_t \equiv \log A_t$ follows a random walk with drift,

$$a_t = a_{t-1} + g + \varepsilon_t, \tag{33}$$

while population is constant:

$$N_t = N. \tag{34}$$

Households use up some of the purchased good in producing inventory management services, earning zero profits in equilibrium. Hence household consumption is given by $S_t - X_t$. With no private storage or capital, the household budget constraint is

$$S_t = W_t N_t + X_t + \Pi_t \tag{35}$$

for Π_t the household's share of the profits of the representative firm. Note that (34) is the condition for labor market equilibrium and output equilibrium is then assured by Walras' Law.

Consider first the case of deterministic growth ($E(\varepsilon_t^2) = 0$). It follows from (26)-(35) that along the steady-state growth path, M_t is constant while the log of each element of $\mathbf{Z}_t = (W_t, Q_t, H_t, S_t, X_t, \Pi_t, A_t)'$ grows linearly at the rate g.

With stochastic growth ($E(\varepsilon_t^2) > 0$), from the standard log linearization around the steady-state growth path we have the familiar result that each element of $\log \mathbf{Z}_t$ is individually I(1) and any two elements of $\log \mathbf{Z}_t$ are cointegrated with cointegrating vector $(1, -1)'$; see for example King, Plosser, Stock, and Watson (1991).

One could also use a linear as opposed to log-linear approximation to the stochastic growth path, for example, treating $H_t - H_{t-1}$ rather than $\log(H_t) - \log(H_{t-1})$ as I(0). To do so, substitute (26) into (29),

$$M_t = \frac{1}{\alpha}\left(\frac{Q_t}{B_t}\right)^{(1/\alpha)-1}, \tag{36}$$

where

$$B_t \equiv A_t^{1/(1-\alpha)} W_t^{-\alpha/(1-\alpha)}.$$

Note that Q_t and B_t are both $O_p(A_t)$ along the stationary growth path. Taking a first-order Taylor approximation to (36) around $(Q_0, B_0)'$ and rearranging,

$$M_t \simeq M_0 + \left(\frac{1}{\alpha} - 1\right)\frac{M_0}{Q_0}[Q_t - (Q_0/B_0)B_t]. \tag{37}$$

Recall that along the stationary growth path, $\log(Q_t)$ and $\log(B_t)$ are both I(1) while $\log(Q_t) - \log(B_t)$ is stationary with mean $\log(Q_0/B_0)$. In the linear approximation to this path, Q_t and B_t are regarded as I(1), and must be cointegrated with cointegrating vector $(1, -(Q_0/B_0))'$ given the stationarity of M_t. Defining $a_1 \equiv [(1/\alpha) - 1]M_0/Q_0$ and $U_{ct} \equiv (Q_0/B_0)B_t$, expression (37) can be written

$$M_t \simeq M_0 + a_1(Q_t - U_{ct}). \tag{38}$$

We can likewise approximate

$$\gamma\left(\frac{H_t}{S_{t+1}} - \xi\right) \simeq k_0 + a_1 a_2 (H_t - a_3 S_{t+1}) \tag{39}$$

where $k_0 \equiv \gamma[(H_0/S_1) - \xi]$, $a_2 \equiv \gamma/(S_1 a_1)$, and $a_3 \equiv (H_0/S_1)$. Substituting (38) and (39) into (31),

$$a_1(Q_t - U_{ct}) = \beta E_t[a_1(Q_{t+1} - U_{c,t+1}) - a_1 a_2 (H_t - a_4 - a_3 S_{t+1})] \tag{40}$$

for $a_4 \equiv -(\beta a_1 a_2)^{-1}[\beta k_0 + (1-\beta)M_0]$. Equation (40) will be recognized as identical to (4) when $a_0 = 0$. Hence the solution to the linear-quadratic optimization problem (1)-(3) can be motivated as a linear approximation to the general equilibrium stochastic growth path implied by (28).

Recall from (38) that $Q_t - U_{ct}$ must be stationary in this equilibrium. Subtracting U_{ct} from (3) results in

$$Q_t - U_{ct} = S_t - U_{ct} + (H_t - H_{t-1}).$$

Since $H_t - H_{t-1}$ is also regarded as stationary for purposes of this approximation, it follows that $(S_t, U_{ct})'$ must be cointegrated with cointegrating vector $(1, -1)'$.

Of course, if growth is actually exponential as the model implies (and as most economic time series appear to exhibit), a log-linear approximation would be superior to the linear approximation analyzed above. Viewing (36) as a function of $(q_t, b_t)' \equiv (\log(Q_t), \log(B_t))'$, the Taylor approximation would be

$$M_t \simeq M_0 \left[1 - \left(\frac{1}{\alpha} - 1\right)(q_0 - b_0)\right] + M_0 \left(\frac{1}{\alpha} - 1\right)(q_t - b_t). \tag{41}$$

Note that $(q_t, b_t)'$ must be cointegrated with cointegrating vector $(1, -1)'$. Similarly approximating the left-hand side of (39) with a linear function of $(h_t, s_{t+1})' \equiv (\log(H_t), \log(S_{t+1}))'$ gives

$$\gamma\left(\frac{H_t}{S_{t+1}} - \xi\right) \simeq \gamma[(H_0/S_1)(1 - h_0 + s_1) - \xi] + \gamma(H_0/S_1)(h_t - s_{t+1}). \tag{42}$$

Substituting (41) and (42) into (31) and dividing both sides by $M_0[(1/\alpha) - 1]$ yields

$$q_t - u_{ct} = \beta E_t[(q_{t+1} - u_{c,t+1}) - a_2^*(h_t - a_4^* - s_{t+1})] \tag{43}$$

where $u_{ct} \equiv b_t$. Equation (43) will again be recognized as an expression of the form of (4) in which the variables Q_t, H_t, and S_t have simply been replaced by their natural logarithms. Hence, if the goal is to fit the model using only the Euler equation (4), one might want to use logarithms rather than levels of the variables in order to better capture exponential as opposed to linear growth. Again obviously $(s_t, u_{ct})'$ are cointegrated with cointegrating vector $(1, -1)'$.

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