Answer key for the final exam in 2020

1a.) $\hat{\boldsymbol{\beta}} = (\mathbf{X}'\mathbf{V}^{-1}\mathbf{X})^{-1}(\mathbf{X}'\mathbf{V}^{-1}\mathbf{y})$

$\hat{\sigma}^2 = (T-k)^{-1}(\mathbf{y}-\mathbf{X}\hat{\boldsymbol{\beta}})'\mathbf{V}^{-1}(\mathbf{y}-\mathbf{X}\hat{\boldsymbol{\beta}})$

$\tau = \hat{\beta}_1/\sqrt{\hat{\sigma}^2\mathbf{e}_1'(\mathbf{X}'\mathbf{V}^{-1}\mathbf{X})^{-1}\mathbf{e}_1}$ for $\mathbf{e}_1$ first column of $\mathbf{I}_k$

τ has an exact small-sample Student t distribution with $T-k$ degrees of freedom under the stated assumptions

1b.) $h = \left[\mathbf{g}(\hat{\boldsymbol{\beta}})\right]'\left[\hat{\sigma}^2\mathbf{H}(\boldsymbol{\beta}_0)(\mathbf{X}'\mathbf{V}^{-1}\mathbf{X})^{-1}\mathbf{H}(\boldsymbol{\beta}_0)'\right]^{-1}\mathbf{g}(\hat{\boldsymbol{\beta}})$

$\mathbf{H}(\boldsymbol{\beta}_0) = \frac{\partial\mathbf{g}}{\partial\boldsymbol{\beta}'}|_{\boldsymbol{\beta}=\boldsymbol{\beta}_0}$

Treat h as approximately $\chi^2(2)$; this is only asymptotic approximation

1c.) (1) White does not do anything with off-diagonal elements of $\mathbf{V}$ but above procedure does; (2) White adjustment to diagonal terms is inefficiently estimated and only approximate; (3) White uses OLS estimate $\mathbf{b}$ which is inefficient.

2.) plim $b = [E(x_t^2)]^{-1}[E(x_t y_t)]$ where

$E(x_t y_t) = E[x_t(g(x_t)+u_t)] = E[x_t g(x_t)]$

3a.) need $r = 2$ since we want to estimate both β_0 and β_1. The first element of the (2×1) vector $\mathbf{x}_t$ would usually be a constant term

3b.)Assuming that a constant term is first element of $\mathbf{x}_t$, regress z_t on $\mathbf{x}_t$. Do OLS F-test that coefficients other than constant term are all zero (we don't want to test that all coefficients are zero, since this is basically just testing whether z_t and x_t have mean zero). If F-statistic is less than 10, instruments might be weak.

4a.) $\frac{\partial\ell}{\partial p} = \frac{\sum x_t}{p} - \frac{\sum(n-x_t)}{1-p} = \frac{\left[(1-p)\sum x_t\right]-\left[p\sum(n-x_t)\right]}{p(1-p)} = \frac{\left[\sum x_t\right]-pTn}{p(1-p)}$.

Setting this to zero gives $\hat{p}_{MLE} = (Tn)^{-1}\sum x_t$

4b.) Notice $\hat{p}_{MLE}$ is n^{-1} times the sample mean of T i.i.d. variables, each with variance $np(1-p)$. Thus

$\sqrt{T}(\hat{p}-p_0) \xrightarrow{L} n^{-1}N(0,np(1-p)) \sim N(0,p(1-p)/n)$.

No additional assumptions needed.

4c.) $h(p,x_t) = \frac{\partial\ell_t}{\partial p} = \frac{x_t}{p} - \frac{(n-x_t)}{(1-p)}$

$E[h(p,x_t)] = \frac{np_0}{p} - \frac{(n-np_0)}{(1-p)} = \frac{(1-p)np_0-pn(1-p_0)}{p(1-p)} = \frac{n(p_0-p)}{p(1-p)}$ which equals 0 at $p = p_0$ and not zero at all other p

4di) Since $r = a = 1$, $\hat{p}_{GMM}$ satisfies

$T^{-1}\sum h(\hat{p},x_t) = 0$ or $\frac{T^{-1}\sum x_t}{\hat{p}} - \frac{T^{-1}\sum(n-x_t)}{1-\hat{p}} = 0$, same FOC as MLE. Thus $\hat{p}_{MLE} = \hat{p}_{GMM}$.

4dii) $\hat{\mathbf{D}}' = T^{-1}\left.\frac{\sum\partial h(p,x_t)}{\partial p}\right|_{p=\hat{p}} = T^{-1}\left[-\frac{\sum x_t}{\hat{p}^2} - \frac{\sum(n-x_t)}{(1-\hat{p})^2}\right] = -\frac{n\hat{p}}{\hat{p}^2} - \frac{n(1-\hat{p})}{(1-\hat{p})^2} = -\frac{n}{\hat{p}} - \frac{n}{(1-\hat{p})}$

4diii) $\hat{\mathbf{D}}' \xrightarrow{p} -\frac{n}{p_0} - \frac{n}{(1-p_0)} = -\frac{n}{p_0(1-p_0)}$

4div) $\hat{\mathbf{S}} = T^{-1}\sum[h(\hat{p},x_t)^2 = T^{-1}\sum\left[\frac{x_t}{\hat{p}} - \frac{(n-x_t)}{(1-\hat{p})}\right]^2 = \frac{T^{-1}}{\hat{p}^2(1-\hat{p})^2}\sum(x_t-n\hat{p})^2$