Answer key for the final exam in 2018

1.) The condition is always true (even if $\mathbf{X}$ has rank less than k)

2a.)

$$p_t = \frac{u_t^d - u_t^s}{\gamma - \beta} \quad q_t = \frac{\gamma}{\gamma - \beta} u_t^d - \frac{\beta}{\gamma - \beta} u_t^s$$

2b.) valid if $E(u_t^s u_t^d) = 0$, i.e., valid if supply shock uncorrelated with demand shock

relevant if $E\left(u_t^s \frac{u_t^d - u_t^s}{\gamma - \beta}\right) \neq 0$. If we assume valid, this is true provided $E(u_t^s)^2 > 0$, i.e. as long as there are some supply shocks

2c.)

$$\hat{\beta}_{IV} = \beta + \frac{\sum u_t^s u_t^d}{\sum u_t^s p_t}$$

$$\sqrt{T}(\hat{\beta}_{IV} - \beta) = \frac{\sqrt{T}\,T^{-1} \sum u_t^s u_t^d}{T^{-1} \sum u_t^s p_t}.$$

Numerator is $\sqrt{T}$ times the sample mean of an i.i.d. variable with mean 0 and variance $\sigma_s^2 \sigma_d^2$ which converges to $N(0, \sigma_s^2 \sigma_d^2)$ under the assumptions in (2b) under CLT. Denominator is sample mean of i.i.d. variable which converges in probability to $E\left(u_t^s \frac{u_t^d - u_t^s}{\gamma - \beta}\right) = -\sigma_s^2/(\gamma - \beta)$. Hence

$$\sqrt{T}(\hat{\beta}_{IV} - \beta) \xrightarrow{L} N\left(0, \frac{\sigma_d^2(\gamma - \beta)^2}{\sigma_s^2}\right).$$

No additional assumptions are necessary.

3d.) Estimate by OLS

$$q_t = \alpha(q_t - \gamma_0 p_t) + v_t$$

and calculate standard t-test of $H_0 : \alpha = 0$.

3a.)

$$b = \beta + \frac{\sum x_t u_t}{\sum x_t^2}$$

$$\sqrt{T}(b - \beta) = \frac{\sqrt{T}\,T^{-1} \sum x_t u_t}{T^{-1} \sum x_t^2}.$$

Numerator converges in distribution to a Normal variable with mean zero and variance $E(x_t^2 u_t^2) = Q[\pi\sigma_1^2 + (1 - \pi)\sigma_0^2]$. Denominator converges in probability to Q. Hence $\sqrt{T}(b - \beta) \xrightarrow{L} N(0, Q^{-1}[\pi\sigma_1^2 + (1 - \pi)\sigma_0^2])$.

3b.)

$$\frac{\partial \ell}{\partial \beta} = \frac{\sum (y_t - x_t\beta)x_t z_t}{\sigma_1^2} + \frac{\sum (y_t - x_t\beta)x_t(1-z_t)}{\sigma_0^2}$$
$$= \sum (\tilde{y}_t - \tilde{x}_t\beta)\tilde{x}_t$$

for

$$\tilde{y}_t = \frac{y_t z_t}{\sigma_1} + \frac{y_t(1-z_t)}{\sigma_0} \qquad \tilde{x}_t = \frac{x_t z_t}{\sigma_1} + \frac{x_t(1-z_t)}{\sigma_0}.$$

Hence $\hat{\beta}_{MLE} = \left(\sum \tilde{x}_t^2\right)^{-1}\left(\sum \tilde{x}_t \tilde{y}_t\right),$ also known as the GLS estimate.

3c.) Notice $\tilde{y}_t = \tilde{x}_t\beta + \tilde{u}_t$ for $\tilde{u}_t = \frac{u_t z_t}{\sigma_1} + \frac{u_t(1-z_t)}{\sigma_0}$ with $\tilde{u}_t|z_t \sim N(0,1)$ and thus $\tilde{u}_t \sim N(0,1)$. Hence

$$\sqrt{T}\,(\hat{\beta}_{MLE} - \beta) = \frac{\sqrt{T}\,T^{-1}\sum \tilde{x}_t\tilde{u}_t}{T^{-1}\sum \tilde{x}_t^2}.$$

The denominator converges in probability to $E(\tilde{x}_t^2) = \tilde{Q}$ where we can calculate $\tilde{Q}$ from

$$E(\tilde{x}_t^2|z_t) = \frac{E(x_t^2)z_t}{\sigma_1^2} + \frac{E(x_t^2)(1-z_t)}{\sigma_0^2}$$
$$E(\tilde{x}_t^2) = Q\left[\frac{\pi}{\sigma_1^2} + \frac{1-\pi}{\sigma_0^2}\right].$$

The numerator converges to $N(0,\tilde{Q})$. Hence $\sqrt{T}\,(\hat{\beta}_{MLE} - \beta) \xrightarrow{L} N(0,\tilde{Q}^{-1})$. The ratio of the variance of b to the variance of $\hat{\beta}_{MLE}$ is

$$\frac{Q^{-1}[\pi\sigma_1^2 + (1-\pi)\sigma_0^2]}{Q^{-1}\left[\frac{\pi}{\sigma_1^2} + \frac{1-\pi}{\sigma_0^2}\right]^{-1}} = \left[\frac{\pi}{\sigma_1^2} + \frac{1-\pi}{\sigma_0^2}\right][\pi\sigma_1^2 + (1-\pi)\sigma_0^2]$$

which from the hint is bigger than 1. It is not surprising that the MLE or GLS estimator has a smaller asymptotic variance than OLS.

e.)

$$H = \frac{\partial^2 \ell}{\partial \beta^2} = -\sum \tilde{x}_t^2$$

so $T^{-1}H \xrightarrow{p} -E(\tilde{x}_t^2) = -\tilde{Q}$. That -1 times the inverse of this is equal to the asymptotic variance of $\hat{\beta}_{MLE}$ is the usual expected result.

f.)

$$\frac{\partial \ell}{\partial \sigma_1^2} = -\frac{T_1}{2\sigma_1^2} + \frac{\sum (y_t - x_t\beta)^2 z_t}{2\sigma_1^4}.$$

Setting to zero gives

$$\begin{aligned}
\hat{\sigma}_1^2 &= T_1^{-1} \sum (y_t - x_t\hat{\beta})^2 z_t \\
&= T_1^{-1} \left\{ \sum (y_t - x_t\beta + x_t\beta - x_t\hat{\beta})^2 z_t \right\} \\
&= T_1^{-1} \sum (y_t - x_t\beta)^2 z_t + 2(\beta - \hat{\beta}) T_1^{-1} \sum u_t x_t z_t + (\beta - \hat{\beta})^2 T_1^{-1} \sum x_t^2 z_t.
\end{aligned}$$

The first term converges in probability to σ_1^2 and the second two terms converge in probability to 0 from consistency of $\hat{\beta}$.

g.) For $\hat{u}_t = y_t - x_t b$ the OLS residuals, regress $\hat{u}_t^2 = \alpha_0 + \alpha_1 z_t + v_t$. The R^2 of this regression times the sample size T should be asymptotically $\chi^2(1)$ under H_0. In other words, reject H_0 if $TR^2 > 3.84$.

4a.) Use $\hat{\mathbf{\Gamma}}_0$ where $\hat{\mathbf{\Gamma}}_v = T^{-1} \sum_{t=v+1}^{T} (\mathbf{y}_t - \bar{\mathbf{y}})(\mathbf{y}_{t-v} - \bar{\mathbf{y}})'$.

4b.) Use $T^{-1}\hat{\mathbf{S}}$ where

$$\hat{\mathbf{S}} = \hat{\mathbf{\Gamma}}_0 + \sum_{v=1}^{q} \left[1 - \frac{q}{v+1} \right] (\hat{\mathbf{\Gamma}}_v + \hat{\mathbf{\Gamma}}_v')$$

for some q.