Answer key for the final exam in 2017

1a.) (v)
1b.) (vii)
1c.) (v) and (vi)
1d.) none
1e.) none

2a.) $\hat{\alpha}_{OLS} = (\sum x_{1t}^2)^{-1}(\sum x_{1t}y_t) = [T^{-1}\sum(x_t^* + v_{1t})^2]^{-1}[T^{-1}\sum(x_t^* + v_{1t})(\beta x_t^* + \varepsilon_t)]$
$\xrightarrow{p} (\omega^2 + \lambda_1^2)^{-1}(\omega^2)\beta$

No other assumptions needed since we have LLN for i.i.d. variables with finite mean. Only consistent if $\lambda_1 = 0$.

2b.) $\hat{\alpha}_{IV} = (\sum x_{2t}x_{1t})^{-1}(\sum x_{2t}y_t) = [T^{-1}\sum(x_t^* + v_{2t})(x_t^* + v_{1t})]^{-1}[T^{-1}\sum(x_t^* + v_{3t})(\beta x_t^* + \varepsilon_t)]$
$\xrightarrow{p} (\omega^2)^{-1}(\omega^2)\beta = \beta$

Consistent as long as $\omega^2 > 0$

2c.) $\text{Corr}(x_{1t}, x_{2t}) = \frac{E(x_t^*+v_{1t})(x_t^*+v_{2t})}{\sqrt{E(x_t^*+v_{1t})^2 E(x_t^*+v_{2t})^2}} = \frac{\omega^2}{\sqrt{(\omega^2+\lambda_1^2)(\omega^2+\lambda_2^2)}}$

Instruments are strong when this is big, that is, when λ_1 and λ_2 are small.

2d.) Use OLS t-test of $\gamma = 0$ in regression $\tilde{y}_t = \gamma x_{2t} + u_t$ for $\tilde{y}_t = y_t - 2x_{1t}$, namely $\tau = \hat{\gamma}/\sqrt{s^2/\sum x_{2t}^2}$ for $\hat{\gamma} = [\sum(x_{2t}^2)]^{-1}[\sum x_{2t}(y_t - 2x_{1t})]$ and $s^2 = (T-1)^{-1}\sum(y_t - 2x_{1t} - \hat{\gamma}x_{2t})^2$. The statistic τ has an exact Student t distribution with $T-1$ degrees of freedom under the assumptions stated. Reject H_0 if the statistic is larger in absolute value than the 5% critical value.

3a.) $\ell(\lambda) = -T\log\lambda + \sum_{t=1}^T \log y_t - \sum_{t=1}^T \frac{y_t^2}{2\lambda}$
$\frac{\partial\ell(\lambda)}{\partial\lambda} = \frac{-T}{\lambda} + \sum_{t=1}^T \frac{y_t^2}{2\lambda^2} \Rightarrow \hat{\lambda} = T^{-1}\sum_{t=1}^T \frac{y_t^2}{2}$

3b.) $s_t = \frac{-1}{\lambda} + \frac{y_t^2}{2\lambda^2} \quad E(s_t) = \frac{-1}{\lambda} + \frac{2\lambda}{2\lambda^2} = 0$

3ci.) $\mathbf{h}(\boldsymbol{\theta}, \mathbf{w}_t) = s_t = \frac{-1}{\lambda} + \frac{y_t^2}{2\lambda^2}$

3cii.) Since i.i.d., $\mathbf{S} = E(s_t^2) = \frac{1}{\lambda^2} - \frac{4\lambda}{2\lambda^3} + \frac{8\lambda^2}{4\lambda^4} = \frac{1}{\lambda^2}$

3ciii.) $\mathbf{D}' = E\left[\frac{1}{\lambda^2} - \frac{y_t^2}{\lambda^3}\right] = \frac{1}{\lambda^2} - \frac{2\lambda}{\lambda^3} = -\frac{1}{\lambda^2}$

3d.) $\sqrt{T}(\hat{\lambda}_{MLE} - \lambda) \xrightarrow{L} N(0, V)$ for $V = (1/\lambda^2)^{-1} = \lambda^2$.

3e.) Note $\hat{\lambda}_{MLE}$ in (a) is sample mean of the i.i.d. random variable $y_t^2/2$ whose expectation is $E(y_t^2)/2 = \lambda$ and whose variance is $E(y_t^4)/4 - \lambda^2 = 2\lambda^2 - \lambda^2 = \lambda^2$ so we have immediately from the CLT that $\sqrt{T}(\hat{\lambda}_{MLE} - \lambda) \xrightarrow{L} N(0, \lambda^2)$