Answer key for the final exam in 2015

1a.) Yes. People in poorer countries may have less access to birth control and more incentives to have children to compensate for infant mortality and have someone to provide for parents in old age. Thus y_i could be causing x_i rather than the other way around, biasing the coefficient to be more negative.

b.) Look for some exogenous source of variation in government provision of birth control as a possible instrument.

2a.) $\mathbf{b} = (\mathbf{X}'\mathbf{X})^{-1}\mathbf{X}'\mathbf{y} \quad s^2 = (T-2)^{-1}(\mathbf{y}-\mathbf{Xb})'(\mathbf{y}-\mathbf{Xb})$

b.) $(b_1 - b_2^2)/s\sqrt{\mathbf{g}'(\mathbf{X}'\mathbf{X})^{-1}\mathbf{g}}$ for $\mathbf{g} = (1, -2b_2)'$. This is asymptotically $N(0,1)$ (not exact Student t because it's only an asymptotic approximation).

c.) (i) Estimate by unrestricted OLS and see how high the log likelihood is unrestricted. This turns out to be

$$(-T/2)(1+\log 2\pi) - (T/2)\log\hat{\sigma}^2$$

$$\hat{\sigma}^2 = T^{-1}\sum_{t=1}^{T}(y_t - x_{t1}b_1 - x_{t2}b_2)^2.$$

(ii) Maximize the restricted log likelihood, whose maximized value turns out to be

$$(-T/2)(1+\log 2\pi) - (T/2)\log\tilde{\sigma}^2$$

$$\tilde{\sigma}^2 = T^{-1}\sum_{t=1}^{T}(y_t - x_{t1}\tilde{b} - x_{t2}\tilde{b}^2)^2$$

$$\tilde{b} = \arg\min_{\beta}\sum_{t=1}^{T}(y_t - x_{t1}\beta - x_{t2}\beta^2)^2.$$

(iii) Calculate twice the difference of the log likelihood, namely

$$q = T[\log\tilde{\sigma}^2 - \log\hat{\sigma}^2].$$

(iv) Reject H_0 if q is bigger than 5% critical value for a $\chi^2(1)$ (namely, reject if $q > 3.84$).

3a.) Consistent means $\hat{\boldsymbol{\beta}}_{2SLS} \xrightarrow{p} \boldsymbol{\beta}$. Need also rank $\boldsymbol{\Sigma}_{\mathbf{zx}} = k$ and rank $\boldsymbol{\Sigma}_{\mathbf{xx}} = k$. Substituting $y_t = \mathbf{z}_t'\boldsymbol{\beta}$ into formula shows

$$\begin{aligned}
\hat{\boldsymbol{\beta}}_{2SLS} &= \boldsymbol{\beta} + \left[\left(\sum_{t=1}^{T}\mathbf{z}_t\mathbf{x}_t'\right)\left(\sum_{t=1}^{T}\mathbf{x}_t\mathbf{x}_t'\right)^{-1}\left(\sum_{t=1}^{T}\mathbf{x}_t\mathbf{z}_t'\right)\right]^{-1} \times \\
&\quad \left[\left(\sum_{t=1}^{T}\mathbf{z}_t\mathbf{x}_t'\right)\left(\sum_{t=1}^{T}\mathbf{x}_t\mathbf{x}_t'\right)^{-1}\left(\sum_{t=1}^{T}\mathbf{x}_t u_t\right)\right] \\
&= \boldsymbol{\beta} + \left[\left(T^{-1}\sum_{t=1}^{T}\mathbf{z}_t\mathbf{x}_t'\right)\left(T^{-1}\sum_{t=1}^{T}\mathbf{x}_t\mathbf{x}_t'\right)^{-1}\left(T^{-1}\sum_{t=1}^{T}\mathbf{x}_t\mathbf{z}_t'\right)\right]^{-1} \times \\
&\quad \left[T^{-1}\left(\sum_{t=1}^{T}\mathbf{z}_t\mathbf{x}_t'\right)\left(T^{-1}\sum_{t=1}^{T}\mathbf{x}_t\mathbf{x}_t'\right)^{-1}\left(T^{-1}\sum_{t=1}^{T}\mathbf{x}_t u_t\right)\right] \\
&\xrightarrow{p} \boldsymbol{\beta} + [\boldsymbol{\Sigma}_{\mathbf{zx}}\boldsymbol{\Sigma}_{\mathbf{xx}}^{-1}\boldsymbol{\Sigma}_{\mathbf{xz}}]^{-1}[\boldsymbol{\Sigma}_{\mathbf{zx}}\boldsymbol{\Sigma}_{\mathbf{xx}}^{-1}E(\mathbf{x}_t u_t)]
\end{aligned}$$

which equals $\boldsymbol{\beta}$ under the mds assumption.

b.) The CLT gives us $\left(T^{-1/2}\sum_{t=1}^{T}\mathbf{x}_t u_t\right) \xrightarrow{L} \mathbf{q} \sim N(\mathbf{0}, \mathbf{S})$ and

$$\sqrt{T}(\hat{\boldsymbol\beta}_{2SLS}-\boldsymbol\beta)=\left[\left(T^{-1}\sum_{t=1}^{T}\mathbf{z}_t\mathbf{x}_t'\right)\left(T^{-1}\sum_{t=1}^{T}\mathbf{x}_t\mathbf{x}_t'\right)^{-1}\left(T^{-1}\sum_{t=1}^{T}\mathbf{x}_t\mathbf{z}_t'\right)\right]^{-1}\times$$

$$\left[T^{-1}\left(\sum_{t=1}^{T}\mathbf{z}_t\mathbf{x}_t'\right)\left(T^{-1}\sum_{t=1}^{T}\mathbf{x}_t\mathbf{x}_t'\right)^{-1}\left(T^{-1/2}\sum_{t=1}^{T}\mathbf{x}_tu_t\right)\right]$$

$$\xrightarrow{L}\mathbf{Aq}$$

for

$$\mathbf{A}=[\boldsymbol\Sigma_{\mathbf{zx}}\boldsymbol\Sigma_{\mathbf{xx}}^{-1}\boldsymbol\Sigma_{\mathbf{xz}}]^{-1}[\boldsymbol\Sigma_{\mathbf{zx}}\boldsymbol\Sigma_{\mathbf{xx}}^{-1}].$$

Thus

$$\sqrt{T}(\hat{\boldsymbol\beta}_{2SLS}-\boldsymbol\beta)\xrightarrow{L}N(\mathbf{0},\mathbf{ASA}').$$

c.)

$$\hat{\mathbf{S}}=T^{-1}\sum_{t=1}^{T}\hat{u}_t^2\mathbf{x}_t\mathbf{x}_t'$$

$$\hat{u}_t=(y_t-\mathbf{z}_t'\hat{\boldsymbol\beta}_{2SLS})$$

$$\hat{\mathbf{A}}=\left[\left(T^{-1}\sum_{t=1}^{T}\mathbf{z}_t\mathbf{x}_t'\right)\left(T^{-1}\sum_{t=1}^{T}\mathbf{x}_t\mathbf{x}_t'\right)^{-1}\left(T^{-1}\sum_{t=1}^{T}\mathbf{x}_t\mathbf{z}_t'\right)\right]^{-1}\times$$

$$\left[T^{-1}\left(\sum_{t=1}^{T}\mathbf{z}_t\mathbf{x}_t'\right)\left(T^{-1}\sum_{t=1}^{T}\mathbf{x}_t\mathbf{x}_t'\right)^{-1}\right].$$

4a.) $\mathcal{L}=\sum_{t=1}^{T}[\log\lambda-\lambda y_t]$

b.) $\sum_{t=1}^{T}[(1/\lambda)-y_t]=0\Rightarrow\hat\lambda=1/\bar{y}$ for $\bar{y}=T^{-1}\sum_{t=1}^{T}y_t$.

c.) $h(\theta,w_t)=(1/\lambda)-y_t$.

d.) $S=E[h(\theta,w_t)]^2=E(y_t-\lambda^{-1})^2=\lambda^{-2}$

$\partial h/\partial\theta=-\lambda^{-2}\Rightarrow\hat{D}=-\hat\lambda^{-2}$

e.) $\hat{V}=(\hat{D}\hat{S}^{-1}D)^{-1}=[-\hat\lambda^{-2}\hat\lambda^2(-\hat\lambda^{-2})]^{-1}=\hat\lambda^2$

$\sqrt{T}(\hat\lambda-\lambda_0)\xrightarrow{L}N(0,\lambda_0^2)$

f.) One estimate we could use is

$$\hat\gamma_0=T^{-1}\sum_{t=1}^{T}(y_t-\bar{y})^2$$

$$\hat\gamma_1=T^{-1}\sum_{t=2}^{T}(y_t-\bar{y})(y_{t-1}-\bar{y})$$

$$\hat{S}=\hat\gamma_0+2\hat\gamma_1$$

$$\hat{V}=\hat{S}\hat\lambda^4.$$