Answer key for the final exam in 2014

1a.) Yes, this could be completely spurious regression. To test, estimate by OLS $\hat{u}_t = \rho_0 + \rho_1 G_t + \rho_2 \hat{u}_{t-1}$ for $\hat{u}_t$ the OLS residual from the first regression. If TR^2 from this second regression is bigger than 3.84 (the 5% critical value for $\chi^2(1)$), this is evidence of serial correlation and potential spurious regression. Fix by adding lags to the regression

$$y_t = \gamma_0 + \gamma_1 G_t + \gamma_2 y_{t-1} + \gamma_3 G_{t-1} + u_t.$$

b.) Yes. When productivity and population grow, government will spend more. Thus both y_t and G_t increase because of the economic growth, not from effect of G_t on y_t. This positive correlation between u_t and G_t would bias the estimate of β upward.

2a.) For $\hat{u}_t = y_t - \mathbf{b}'\mathbf{x}_t$ regress $\hat{u}_t^2 = \alpha_0 + \alpha_1 x_{t1} + v_t$. If TR^2 from this regression is bigger than 3.84 (the 5% critical value for $\chi^2(1)$), that is evidence that $\alpha_1 \neq 0$.

b.) Use White formula $(\mathbf{X}'\mathbf{X})^{-1}\left(\sum_{t=1}^{T} \hat{u}_t^2 \mathbf{x}_t\mathbf{x}_t'\right)(\mathbf{X}'\mathbf{X})^{-1}$.

c.) $\hat{\boldsymbol{\beta}} = (\tilde{\mathbf{X}}'\tilde{\mathbf{X}})^{-1}\tilde{\mathbf{X}}'\tilde{\mathbf{y}}$ for $\tilde{y}_t = y_t/\sqrt{\hat{\alpha}_0 + \hat{\alpha}_1 x_{t1}}$, $\tilde{\mathbf{x}}_t = \mathbf{x}_t\sqrt{\hat{\alpha}_0 + \hat{\alpha}_1 x_{t1}}$ and $\hat{\alpha}_0$ and $\hat{\alpha}_1$ the coefficients from the regression in (2a).

3a.) $\mathbf{h}(\theta, \mathbf{w}_t) = \mathbf{x}_t[y_t - q(z_t, \theta)]$

$g(\theta, \mathbf{Y}_T) = T^{-1}\sum_{t=1}^{T} \mathbf{x}_t[y_t - q(z_t, \theta)]$

b.)

$$\min_{\theta}\; T^{-1}\left\{\sum_{t=1}^{T}[y_t - q(\theta, z_t)]\mathbf{x}_t'\right\}[\sigma^2\mathbf{Q}]^{-1}\left\{\sum_{t=1}^{T}\mathbf{x}_t[y_t - q(\theta, z_t)]\right\}$$

$$\left\{\sum_{t=1}^{T} \underset{(1\times 1)}{q'(\hat{\theta}, z_t)}\; \underset{(1\times m)}{\mathbf{x}_t'}\right\}\left[\underset{(m\times m)}{\sigma^2\mathbf{Q}}\right]^{-1}\left\{\sum_{t=1}^{T} \underset{(m\times 1)}{\mathbf{x}_t}\; \underset{(1\times 1)}{[y_t - q(\hat{\theta}, z_t)]}\right\} = \underset{(1\times 1)}{0}$$

c.)

$$\left\{\sum_{t=1}^{T} \underset{(1\times 1)}{z_t}\underset{(1\times m)}{\mathbf{x}_t'}\right\}\left[\underset{(m\times m)}{\sigma^2\mathbf{Q}}\right]^{-1}\left\{\sum_{t=1}^{T} \underset{(m\times 1)}{\mathbf{x}_t}\; \underset{(1\times 1)}{[y_t - z_t\hat{\theta}]}\right\} = \underset{(1\times 1)}{0}$$

$$\hat{\theta} = \left\{\left(\sum_{t=1}^{T} z_t\mathbf{x}_t'\right)[\sigma^2\mathbf{Q}]^{-1}\left(\sum_{t=1}^{T}\mathbf{x}_t z_t\right)\right\}^{-1} \times \left\{\left(\sum_{t=1}^{T} z_t\mathbf{x}_t'\right)[\sigma^2\mathbf{Q}]^{-1}\left(\sum_{t=1}^{T}\mathbf{x}_t y_t\right)\right\}$$

d.) The following $(m \times 1)$ vector should have rank 1,

$$\text{plim}\; T^{-1}\sum_{t=1}^{T} \left.\frac{\partial \mathbf{h}(\theta, \mathbf{w}_t)}{\partial \theta}\right|_{\theta=\theta_0} = E[\mathbf{x}_t q'(\theta_0, \mathbf{w}_t)]$$

in other words, at least one element of $E[\mathbf{x}_t q'(\theta_0, \mathbf{w}_t)]$ must not be zero.

e.) Let $\hat{V} = \left\{\hat{\mathbf{D}}[\sigma^2\mathbf{Q}]^{-1}\hat{\mathbf{D}}'\right\}^{-1}$ for $\hat{\mathbf{D}} = T^{-1}\sum_{t=1}^{T} q'(\hat{\theta}, z_t)\mathbf{x}_t'$

Then t test is $\hat{\theta}/\sqrt{T\hat{V}}$

f.)

$$T^{-1}\left\{\sum_{t=1}^{T}[y_t - q(0, z_t)]\mathbf{x}_t'\right\}[\sigma^2\mathbf{Q}]^{-1}\left\{\sum_{t=1}^{T}\mathbf{x}_t[y_t - q(0, z_t)]\right\} \xrightarrow{L} \chi^2(m)$$

even if instruments weak or irrelevant. Alternative valid answer: regress $\tilde{y}_t = y_t - q(z_t, 0)$ on $\mathbf{x}_t$ by OLS, calculate OLS F test that coefficients are all zero using $T\sigma^2\mathbf{Q}$ in place of OLS $s^2(\mathbf{X}'\mathbf{X})^{-1}$.

4a.) $h(\mu, y_t) = y_t - \mu$ which is i.i.d. $N(0,1)$ under maintained model

b.) $\hat{\mu} = T^{-1}\sum_{t=1}^{T} y_t$

c.) $\partial h(\mu, y_t)/\partial\mu = -1$ so information = 1 and $\sqrt{T}(\hat{\mu} - \mu_0) \xrightarrow{L} N(0,1)$ or standard error for $\hat{\mu}$ is $1/\sqrt{T}$.

d.) $\hat{S} = T^{-1}\sum_{t=1}^{T}(y_t - \hat{\mu})^2 \xrightarrow{p} E(y_t - \mu)^2 = 1.$

e.) μ_0 is value for which $E[h(\mu_0, y_t)] = 0$ or $E(y_t - \mu_0) = 0$ where the expectation is taken with respect to the true distribution of y_t (not necessarily a $N(0,1)$ distribution). In other words, μ_0 is the population mean of y_t, whether or not data are i.i.d. or Gaussian.

f.) $\hat{V} = (\hat{D}\hat{S}^{-1}\hat{D})^{-1} = (1\hat{S}^{-1}1)^{-1} = T^{-1}\sum_{t=1}^{T}(y_t - \hat{\mu})^2$. So quasi-MLE standard error for $\hat{\mu}$ is $\sqrt{\hat{V}/T}$, which is asymptotically equivalent to the standard error we would use if we had not assumed that the variance of y_t was unity.