Answer key for the final exam in 2012

1a.) $\mathbf{b} = (\mathbf{X}'\mathbf{X})^{-1}\mathbf{X}'\mathbf{y}$
b.) $E(\mathbf{b}) = \boldsymbol{\beta}$, yes is unbiased because $E(\boldsymbol{\varepsilon}|\mathbf{X}) = \mathbf{0}$
c.) $\hat{\boldsymbol{\beta}}_{GLS} = (\mathbf{X}'\mathbf{V}^{-1}\mathbf{X})^{-1}\mathbf{X}'\mathbf{V}^{-1}\mathbf{y}$
d.) $\sum(y_t - \mathbf{x}_t'\mathbf{b})^2$ is smaller because OLS minimizes sum of squared residuals
e.) $\mathbf{b}'\mathbf{X}'\mathbf{X}\mathbf{b}/(ks^2)$
f.) No, but $\boldsymbol{\varepsilon}|\mathbf{X} \sim N(0, \sigma^2\mathbf{I}_T)$ would be a sufficient condition

2a.) Positive correlation means biased upwards
b.) $b = (\sum z_t^2)^{-1}(\sum z_t y_t) = \beta + (T^{-1}\sum z_t^2)^{-1}(T^{-1}\sum z_t u_t) \xrightarrow{p} \beta + \sigma_{zz}^{-1}\sigma_{zu}$
c.) relevant: $E(x_t u_t) = 0$;   valid: $E(x_t z_t) \neq 0$
d.) $\hat{\beta}_{2SLS} = (\sum \hat{z}_t^2)^{-1}(\sum \hat{z}_t y_t)$ for $\hat{z}_t = (\sum x_t^2)^{-1}(\sum x_t z_t)x_t$ so
$\hat{\beta}_{2SLS} = [(\sum z_t x_t)(\sum x_t^2)^{-1}(\sum x_t z_t)]^{-1}(\sum z_t x_t)(\sum x_t^2)^{-1}(\sum x_t y_t)$
$= \beta + [(T^{-1}\sum z_t x_t)(T^{-1}\sum x_t^2)^{-1}(T^{-1}\sum x_t z_t)]^{-1}(T^{-1}\sum z_t x_t)(T^{-1}\sum x_t^2)^{-1}(T^{-1}\sum x_t u_t)$
$\xrightarrow{p} \beta + [\sigma_{zx}\sigma_{xx}^{-1}\sigma_{xz}]^{-1}\sigma_{zx}\sigma_{xx}^{-1}0 = \beta$
e.) Regress y_t on x_t by OLS and calculate usual t test that coefficient is zero, that is, calculate $\hat{\pi}/s_\pi$ for
$\hat{\pi} = (\sum x_t^2)^{-1}(\sum x_t y_t)$ $s_\pi^2 = (T-1)^{-1}\sum(y_t - x_t\hat{\pi})^2$. This has exact t distribution with $T-1$ degrees of freedom if $(u_1, \ldots, u_T)'|(x_1, \ldots, x_T)' \sim N(\mathbf{0}, \sigma^2\mathbf{I}_T)$.

3a.) $T/\alpha - \sum \log x_t = 0$ so $\hat{\alpha}_{MLE} = T/\sum \log x_t$
b.) $\boldsymbol{\theta} = \alpha$, $\mathbf{w}_t = x_t$, $\mathbf{h}(\boldsymbol{\theta}, \mathbf{w}_t) = \log x_t - 1/\alpha$
c.) $\hat{\mathbf{D}} = T^{-1}\sum(-1/\hat{\alpha}^2) = -1/\hat{\alpha}^2$ $\mathbf{S} = E[\log x_t - 1/\alpha]^2 = 1/\alpha^2$
$\sqrt{T}(\hat{\alpha}_{MLE} - \alpha) \xrightarrow{L} N(0, V_{MLE})$ $\hat{V}_{MLE} = [(-1/\hat{\alpha}^2)(1/\hat{\alpha}^2)^{-1}(-1/\hat{\alpha}^2)]^{-1} = \hat{\alpha}^2$
d.) $\hat{\alpha}/(\hat{\alpha}-1) = \overline{X}$ so $\hat{\alpha} = \overline{X}/(\overline{X}-1)$
e.) $\frac{\partial h(\alpha, x_t)}{\partial \alpha} = \frac{\alpha}{(\alpha-1)^2} - \frac{1}{(\alpha-1)} = \frac{1}{(\alpha-1)^2}$
$S = E\left[x_t - \frac{\alpha}{\alpha-1}\right]^2 = Var(X_t) = \frac{\alpha}{(\alpha-1)^2(\alpha-2)}$ as given in the exam question
$\sqrt{T}(\hat{\alpha}_{GMM} - \alpha) \xrightarrow{L} N(0, V_{GMM})$
$\hat{V}_{GMM} = \left[\frac{1}{(\hat{\alpha}-1)^2}\frac{(\hat{\alpha}-1)^2(\hat{\alpha}-2)}{\hat{\alpha}}\frac{1}{(\hat{\alpha}-1)^2}\right]^{-1} = \frac{(\hat{\alpha}-1)^2\hat{\alpha}}{(\hat{\alpha}-2)}$
f.) $\frac{V_{GMM}}{V_{MLE}} = \frac{(\alpha-1)^2\alpha/(\alpha-2)}{\alpha^2} = \frac{(\alpha-1)^2}{\alpha(\alpha-2)} > 1$
g.) This is not unexpected; MLE should be asymptotically efficient, GMM need not be