Answer key for the final exam in 2011

1a.) $\begin{bmatrix} a \\ b \end{bmatrix} = \begin{bmatrix} T & \sum x_t \\ \sum x_t & \sum x_t^2 \end{bmatrix}^{-1} \begin{bmatrix} \sum y_t \\ \sum x_t y_t \end{bmatrix}$ is unbiased

b.) Let $\tilde{\mathbf{x}}_t = \begin{bmatrix} \exp(-\gamma x_t/2) \\ x_t \exp(-\gamma x_t/2) \end{bmatrix}$ $\tilde{y}_t = \exp(-\gamma x_t/2) y_t$

$\begin{bmatrix} \hat{\alpha} \\ \hat{\beta} \end{bmatrix} = (\sum \tilde{\mathbf{x}}_t \tilde{\mathbf{x}}_t')^{-1} \sum \tilde{\mathbf{x}}_t \tilde{y}_t$ is unbiased

alternative form: $\begin{bmatrix} \hat{\alpha} \\ \hat{\beta} \end{bmatrix} = (\mathbf{X}'\mathbf{V}^{-1}\mathbf{X})^{-1}\mathbf{X}'\mathbf{V}^{-1}\mathbf{y}$

c.) $\begin{bmatrix} \hat{\alpha} \\ \hat{\beta} \end{bmatrix}$ has smaller variance than $\begin{bmatrix} a \\ b \end{bmatrix}$ and hypothesis tests have correct size with GLS

d.) $\hat{\beta}/\hat{\sigma}_{\hat{\beta}} \sim N(0,1)$ for $\hat{\sigma}_{\hat{\beta}}$ = square root of the $(2,2)$ element of $(\mathbf{X}'\mathbf{V}^{-1}\mathbf{X})^{-1}$ has the exact distribution claimed

e.) Choose a, β, γ to maximize log likelihood

$-(T/2)\log(2\pi) - (1/2)\log|\mathbf{V}| - (1/2)(\mathbf{y} - \mathbf{X}\boldsymbol{\beta})'\mathbf{V}^{-1}(\mathbf{y} - \mathbf{X}\boldsymbol{\beta})$

same statistic as before with $\mathbf{V}$ replaced by $\hat{\mathbf{V}}$ but now $N(0,1)$ is only an approximation

2a.) $\begin{bmatrix} a \\ b \end{bmatrix} = \begin{bmatrix} T & \sum z_t \\ \sum z_t & \sum z_t^2 \end{bmatrix}^{-1} \begin{bmatrix} \sum y_t \\ \sum z_t y_t \end{bmatrix}$

$$= \begin{bmatrix} 1 & T^{-1}\sum(z_t^* + u_t) \\ T^{-1}\sum(z_t^* + u_t) & T^{-1}\sum(z_t^* + u_t)^2 \end{bmatrix}^{-1} \begin{bmatrix} T^{-1}\sum(\alpha + \beta z_t^* + \varepsilon_t) \\ T^{-1}\sum(z_t^* + u_t)(\alpha + \beta z_t^* + \varepsilon_t) \end{bmatrix}$$

$$\xrightarrow{p} \begin{bmatrix} 1 & 0 \\ 0 & (\sigma_z^2 + \sigma_u^2) \end{bmatrix}^{-1} \begin{bmatrix} \alpha \\ \beta\sigma_z^2 \end{bmatrix} = \begin{bmatrix} \alpha \\ \beta\sigma_z^2/(\sigma_z^2 + \sigma_u^2) \end{bmatrix}.$$

b.) $y_t = \alpha + \beta(z_t - u_t) + \varepsilon_t$ so $v_t = \varepsilon_t - \beta u_t$ and $E(x_t v_t) = E(x_t \varepsilon_t) - \beta E(x_t u_t) = 0$.
Relevant if $E(x_t z_t) \neq 0$ and here $E(x_t z_t) = E[x_t(z_t^* + u_t)] = \sigma_{xz}$ so relevant if $\sigma_{xz} \neq 0$

c.) $\begin{bmatrix} \hat{\alpha} \\ \hat{\beta} \end{bmatrix} = \begin{bmatrix} T & \sum z_t \\ \sum x_t & \sum x_t z_t \end{bmatrix}^{-1} \begin{bmatrix} \sum y_t \\ \sum x_t y_t \end{bmatrix}$

$$= \begin{bmatrix} 1 & T^{-1}\sum(z_t^* + u_t) \\ T^{-1}\sum x_t & T^{-1}\sum x_t(z_t^* + u_t) \end{bmatrix}^{-1} \begin{bmatrix} T^{-1}\sum(\alpha + \beta z_t^* + \varepsilon_t) \\ T^{-1}\sum x_t(\alpha + \beta z_t^* + \varepsilon_t) \end{bmatrix}$$

$$\xrightarrow{p} \begin{bmatrix} 1 & 0 \\ 0 & \sigma_{xz} \end{bmatrix}^{-1} \begin{bmatrix} \alpha \\ \beta\sigma_{xz} \end{bmatrix} = \begin{bmatrix} \alpha \\ \beta \end{bmatrix}$$

3a.) If $\varepsilon_i > 0,$ factors other than the observed x's caused country i to have faster growth and rising income levels may have been a factor causing country i to adopt more democratic policies. Hence ε_i and x_{i3} would be positively correlated which would bias $\hat{\beta}_3$ upward.

b.) The plim of the regression is what you'd predict for the growth rate of country i if you already knew the values of x for that country

4a.) $\boldsymbol{\theta} = \boldsymbol{\beta}$ $\mathbf{w}_t = (z_t, \mathbf{x}_t')'$ $\mathbf{h}(\boldsymbol{\theta}, \mathbf{w}_t) = \mathbf{x}_t(y_t - \mathbf{x}_t'\boldsymbol{\beta})$ is $(k \times 1)$

b.) $\hat{\boldsymbol{\theta}} = (\sum \mathbf{x}_t\mathbf{x}_t')^{-1} \sum \mathbf{x}_t y_t$

c.) Note $\boldsymbol{\Gamma}_{-1} = \boldsymbol{\Gamma}_1'$ by definition, so $\mathbf{S} = \boldsymbol{\Gamma}_0 + \boldsymbol{\Gamma}_1 + \boldsymbol{\Gamma}_1'$ $\hat{\mathbf{D}} = -\mathbf{T}^{-1} \sum \mathbf{x}_t\mathbf{x}_t'$ $\hat{\mathbf{V}} = (\hat{\mathbf{D}}\mathbf{S}^{-1}\hat{\mathbf{D}}')^{-1}$

d.) $T(\mathbf{R}\hat{\boldsymbol{\beta}} - \mathbf{r})'[\mathbf{R}\hat{\mathbf{V}}\mathbf{R}']^{-1}(\mathbf{R}\hat{\boldsymbol{\beta}} - \mathbf{r}) \sim \chi^2(m)$

e.) $J = 0$ so cannot be used to test anything