Answer key for the final in 2009

1.) a.) v

b.) v-vi

c.) v-vi

d.) vii

e.) none

f.) none

2.) a.) $\hat{\beta} = \left(\sum_{t=1}^{T} \underline{x}_t \underline{x}_t'\right)^{-1} \left(\sum_{t=1}^{T} \underline{x}_t y_t\right)$ for $\underline{x}_t = (1, x_t)'$.

b.) $\hat{\gamma} = S_{xy}/S_{xx}$

c.)

$$\begin{aligned}
\sum_{t=1}^{T} e_t^2 &= \sum_{t=1}^{T} \left[(y_t - \bar{y}) - \hat{\gamma}(x_t - \bar{x})\right]^2 \\
&= \sum_{t=1}^{T}(y_t - \bar{y})^2 + \sum_{t=1}^{T} \hat{\gamma}^2 (x_t - \bar{x})^2 - 2\hat{\gamma} \sum_{t=1}^{T}(x_t - \bar{x})(y_t - \bar{y}) \\
T^{-1} \sum_{t=1}^{T} e_t^2 &= S_{yy} + \frac{S_{xy}^2}{S_{xx}^2} S_{xx} - 2\frac{S_{xy}^2}{S_{xx}} \\
&= S_{yy} - \frac{S_{xy}^2}{S_{xx}}
\end{aligned}$$

Therefore,

$$\begin{aligned}
R^2 &= 1 - \frac{S_{yy} - \frac{S_{xy}^2}{S_{xx}}}{S_{yy}} \\
&= 1 - 1 + \frac{S_{xy}^2}{S_{xx} S_{yy}} \\
&= r_{xy}^2
\end{aligned}$$

d.)

$$\begin{aligned}
F &= \frac{\hat{\gamma}^2 T S_{xx}}{s^2} = \frac{T S_{xy}^2}{S_{xx}\left(S_{yy} - S_{xy}^2/S_{xx}\right)\frac{T}{T-2}} \\
&= (T-2)\left(\frac{S_{xy}^2}{S_{xx} S_{yy} - S_{xy}^2}\right)
\end{aligned}$$

e.)

$$
\begin{aligned}
F &= (T-2)\left(\frac{1}{-1+\frac{S_{xx}S_{yy}}{S_{xy}^2}}\right) \\
&= (T-2)\left(\frac{1}{-1+1/R^2}\right) \\
&= (T-2)\left(\frac{R^2}{1-R^2}\right)
\end{aligned}
$$

3.) a.) Factors in ε_i such as an economic shock might be positively correlated with s_i. This is expected to result in $\hat{\beta}_2$ being biased upward because if $cov(s_i, \varepsilon_i) > 0$

$$
\hat{\beta}_2 = \beta_2 + \frac{\sum_{i=1}^n (s_i - \bar{s})(\varepsilon_i - \bar{\varepsilon})}{\sum_{i=1}^n (s_i - \bar{s})^2} \xrightarrow{p} \beta_2 + \frac{cov(s_i, \varepsilon_i)}{var(s_i)} > \beta_2
$$

b.) Let $\underline{\mathrm{x}}_i = (1, x_i)'$. $E[\underline{\mathrm{x}}_i(1, s_i)]$ has rank 2.

c.) $Ex_i\varepsilon_i = 0$

d.) Let $\hat{\underline{\mathrm{x}}}_i = (1, \hat{s}_i)'$, $\hat{s}_i = \hat{\alpha}_1 + \hat{\alpha}_2 x_i$ and $\hat{\alpha} = (\hat{\alpha}_1, \hat{\alpha}_2)' = \left(\sum_{i=1}^n \hat{\underline{\mathrm{x}}}_i \hat{\underline{\mathrm{x}}}_i'\right)^{-1} \sum_{i=1}^n \hat{\underline{\mathrm{x}}}_i s_i$.

$$
\hat{\beta} = \left(\sum_{i=1}^n \hat{\underline{\mathrm{x}}}_i \hat{\underline{\mathrm{x}}}_i'\right)^{-1} \sum_{i=1}^n \hat{\underline{\mathrm{x}}}_i y_i,
$$

4.) a.) $E[y_t | y_{t-1}, \ldots, y_1, x_t, \ldots, x_1] = 0$ implies $E[y_t | y_{t-1}, \ldots, y_1] = 0$.

b.) Score $h(\alpha, w_t) = \frac{\partial}{\partial \alpha}[-(1/2)\log(2\pi) - (1/2)\log(\alpha x_t) - y_t^2/(2\alpha x_t)]$.

$$
\begin{aligned}
h(\alpha, w_t) &= \frac{-1}{2\alpha x_t} x_t + \frac{y_t^2}{2\alpha^2 x_t} \\
&= \frac{-1}{2\alpha} + \frac{y_t^2}{2\alpha^2 x_t} \\
E[h(\alpha_0, w_t)] &= \frac{-1}{2\alpha_0} + \frac{1}{2\alpha_0^2} E\left[\frac{y_t^2}{x_t}\right] \\
&= 0
\end{aligned}
$$

c.) $\hat{\alpha} = T^{-1} \sum_{t=1}^T y_t^2 / x_t$

d.) $E[\hat{\alpha}] = \alpha$ and $var[\hat{\alpha}] = E[\hat{\alpha} - E[\hat{\alpha}]]^2 = 2\alpha^2$. Therefore,

$$\sqrt{T}(\hat{\alpha} - \alpha) \xrightarrow{L} N(0, 2\alpha^2)$$

e.) From $E[h(\alpha_0, w_t)] = 0$, GMM estimator $\hat{\alpha}$ make sample moment $T^{-1}\sum_{t=1}^{T} h(\hat{\alpha}, w_t) = 0$. That is,

$$\frac{1}{T}\sum_{t=1}^{T}\left(\frac{-1}{2\alpha} + \frac{y_t^2}{2\hat{\alpha}^2 x_t}\right) = 0$$

f.) As $E[h_t h_s] = 0$ for $t \neq s$,

$$\begin{aligned}
\mathrm{S} &= \sum_{v=-\infty}^{\infty} E\left[h(\alpha_0, w_t)h(\alpha_0, w_{t-v})'\right] \\
&= E\left[h(\alpha_0, w_t)h(\alpha_0, w_t)'\right] \\
&= \frac{1}{4\alpha^2} + \frac{3\alpha^2}{4\alpha^4} - \frac{\alpha}{2\alpha^3} \\
&= \frac{1}{2\alpha^2}
\end{aligned}$$

g.)

$$\begin{aligned}
\mathrm{D}' &= E\left[\frac{\partial h(\alpha, w_t)}{\partial \alpha'}\right] = E\left[\frac{1}{2\alpha^2} - \frac{y_t^2}{\alpha^3 x_t}\right] \\
&= -\frac{1}{2\alpha^2} \\
\mathrm{V} &= \left(\mathrm{D}\mathrm{S}^{-1}\mathrm{D}'\right)^{-1} \\
&= 2\alpha^2
\end{aligned}$$

h.) Yes, because the moment condition $E[h(\alpha_0, w_t)] = 0$ would still hold.

i.) $\hat{\mathrm{V}} = [\hat{\mathrm{D}}\hat{\mathrm{S}}^{-1}\hat{\mathrm{D}}]^{-1}$, where

$$\begin{aligned}
\hat{\mathrm{D}} &= -\frac{1}{2\hat{\alpha}^2} \\
\hat{\mathrm{S}} &= \frac{1}{4\hat{\alpha}^2} + \frac{1}{4\hat{\alpha}^4}\frac{1}{T}\sum_{t=1}^{T}\frac{y_t^4}{x_t^2} - \frac{1}{2\hat{\alpha}^3}\frac{1}{T}\sum_{t=1}^{T}\frac{y_t^2}{x_t} \\
&= \frac{1}{4\hat{\alpha}^4}\frac{1}{T}\sum_{t=1}^{T}\frac{y_t^4}{x_t^2} - \frac{1}{4\hat{\alpha}^2}
\end{aligned}$$