Structural vector autoregressions 2

A. Problem statement

Reduced-form (can easily estimate):

$$\mathbf{y}_t = \mathbf{c} + \mathbf{\Phi}_1 \mathbf{y}_{t-1} + \cdots + \mathbf{\Phi}_p \mathbf{y}_{t-p} + \boldsymbol{\varepsilon}_t$$

$$E(\boldsymbol{\varepsilon}_t \boldsymbol{\varepsilon}_t') = \mathbf{\Omega}$$

$$\frac{\partial \mathbf{y}_{t+s}}{\partial \boldsymbol{\varepsilon}_t'} = \mathbf{\Psi}_s$$

Structural model of interest:

$$\mathbf{B}_0 \mathbf{y}_t = \boldsymbol{\lambda} + \mathbf{B}_1 \mathbf{y}_{t-1} + \cdots + \mathbf{B}_p \mathbf{y}_{t-p} + \mathbf{u}_t$$

$$\boldsymbol{\varepsilon}_t = \mathbf{B}_0^{-1} \mathbf{u}_t$$

$$\frac{\partial \mathbf{y}_{t+s}}{\partial \mathbf{u}_t'} = \frac{\partial \mathbf{y}_{t+s}}{\partial \boldsymbol{\varepsilon}_t'} \frac{\partial \boldsymbol{\varepsilon}_t}{\partial \mathbf{u}_t'} = \mathbf{\Psi}_s \mathbf{B}_0^{-1}$$

Problem: How to estimate $\mathbf{B}_0^{-1}$
(or at least one column of $\mathbf{B}_0^{-1}$)

Example: if

$$\mathbf{B}_0 = \begin{bmatrix} b_0^{(1,1)} & b_0^{(1,2)} & 0 & 0 & 0 \\ b_0^{(2,1)} & b_0^{(2,2)} & 0 & 0 & 0 \\ b_0^{(3,1)} & b_0^{(3,2)} & 1 & 0 & 0 \\ b_0^{(4,1)} & b_0^{(4,2)} & b_0^{(4,3)} & b_0^{(4,4)} & b_0^{(4,5)} \\ b_0^{(5,1)} & b_0^{(5,2)} & b_0^{(5,3)} & b_0^{(5,4)} & b_0^{(5,5)} \end{bmatrix}$$

Then $\widehat{\frac{\partial \mathbf{y}_{t+s}}{\partial u_{3t}}} = \hat{\boldsymbol{\Psi}}_s \hat{p}_{33}^{-1} \hat{\mathbf{p}}_3$

for $\hat{\mathbf{p}}_3$ col 3 and $\hat{p}_{33}$ row 3 col 3 of Cholesky factor $\hat{\boldsymbol{\Omega}} = \hat{\mathbf{P}}\hat{\mathbf{P}}'$

Alternatively, with zero or other restrictions solve

$\hat{\boldsymbol{\Omega}} = \hat{\mathbf{B}}_0^{-1} \hat{\mathbf{D}} (\hat{\mathbf{B}}_0^{-1})'$

Theme today: what alternative strategies are available for identifying $\boldsymbol{\Psi}_s \mathbf{B}_0^{-1}$?

Structural vector autoregressions 2

A. Problem statement
B. Identification using long-run restrictions

x log of productivity
(log GDP minus log civilian labor force)
n log of civilian labor force

$$\mathbf{y}_t = \begin{bmatrix} \Delta x_t \\ \Delta n_t \end{bmatrix} \sim I(0)$$

VAR (reduced-form)

$$\mathbf{y}_t = \mathbf{c} + \mathbf{\Phi}_1 \mathbf{y}_{t-1} + \mathbf{\Phi}_2 \mathbf{y}_{t-2} + \cdots + \mathbf{\Phi}_p \mathbf{y}_{t-p} + \boldsymbol{\varepsilon}_t$$

$$E(\boldsymbol{\varepsilon}_t \boldsymbol{\varepsilon}_t') = \mathbf{\Omega}$$

Structural model:

$$\mathbf{B}_0\mathbf{y}_t = \mathbf{b}_0 + \mathbf{B}_1\mathbf{y}_{t-1} + \mathbf{B}_2\mathbf{y}_{t-2} + \cdots + \mathbf{B}_p\mathbf{y}_{t-p} + \mathbf{u}_t$$

$E(\mathbf{u}_t\mathbf{u}_t') = \mathbf{I}_2$ (normalization)

Relation between representations:

$$\mathbf{u}_t = \mathbf{B}_0\boldsymbol{\varepsilon}_t$$

$$\boldsymbol{\Omega} = \mathbf{B}_0^{-1}(\mathbf{B}_0^{-1})'$$

Premultiply structural model,

$$\mathbf{B}(L)\mathbf{y}_t = \mathbf{b}_0 + \mathbf{u}_t$$

by $\mathbf{C}(L) = \mathbf{B}(L)^{-1}$:

$$\mathbf{y}_t = \boldsymbol{\mu} + \mathbf{C}_0\mathbf{u}_t + \mathbf{C}_1\mathbf{u}_{t-1} + \mathbf{C}_2\mathbf{u}_{t-2} + \cdots$$

which gives structural MA representation

$$\mathbf{u}_t = \begin{bmatrix} u_{1t} \\ u_{2t} \end{bmatrix}$$

u_{1t} technology shock

u_{2t} demand disturbances

Assumption: demand shocks can not have a permanent effect on productivity

$$\lim_{s \to \infty} \frac{\partial x_{t+s}}{\partial u_{2t}} = 0$$

Notice

$$\frac{\partial x_{t+s}}{\partial u_{2t}} = \frac{\partial (x_{t+s} - x_{t+s-1})}{\partial u_{2t}} + \frac{\partial (x_{t+s-1} - x_{t+s-2})}{\partial u_{2t}} + \cdots + \frac{\partial (x_t - x_{t-1})}{\partial u_{2t}}$$

$$\mathbf{y}_t = \begin{bmatrix} x_t - x_{t-1} \\ n_t - n_{t-1} \end{bmatrix}$$

$$\frac{\partial(x_t - x_{t-1})}{\partial u_{2t}} = \frac{\partial y_{1t}}{\partial u_{2t}}$$

$$\mathbf{y}_t = \mu + \mathbf{C}_0 \mathbf{u}_t + \mathbf{C}_1 \mathbf{u}_{t-1}$$

$$+ \mathbf{C}_2 \mathbf{u}_{t-2} + \cdots$$

$$\frac{\partial \mathbf{y}_{t+m}}{\partial \mathbf{u}_t'} = \mathbf{C}_m$$

$$\frac{\partial x_{t+s}}{\partial u_{2t}} = \frac{\partial(x_{t+s} - x_{t+s-1})}{\partial u_{2t}} + \frac{\partial(x_{t+s-1} - x_{t+s-2})}{\partial u_{2t}}$$

$$+ \cdots + \frac{\partial(x_t - x_{t-1})}{\partial u_{2t}}$$

is given by the row 1 column 2 element of

$$\mathbf{C}_0 + \mathbf{C}_1 + \mathbf{C}_2 + \cdots + \mathbf{C}_s$$

$$\lim_{s \to \infty} \frac{\partial x_{t+s}}{\partial u_{2t}} = 0$$

requires that the following matrix is lower triangular:

$$\mathbf{C}_0 + \mathbf{C}_1 + \mathbf{C}_2 + \cdots = \mathbf{C}(1)$$

Goal: find structural disturbances $\mathbf{u}_t$ that are a linear combination of the VAR innovations,

$$\mathbf{u}_t = \mathbf{H}\boldsymbol{\varepsilon}_t,$$

such that:

(1) $E(\mathbf{u}_t\mathbf{u}_t') = \mathbf{I}_2$

$$\Rightarrow \mathbf{H}\boldsymbol{\Omega}\mathbf{H}' = \mathbf{I}_2$$

$$\Rightarrow \boldsymbol{\Omega} = (\mathbf{H}^{-1})(\mathbf{H}^{-1})'$$

(2) $\mathbf{y}_t = \boldsymbol{\mu} + \mathbf{C}(L)\mathbf{u}_t$

(3) $\mathbf{C}(1)$ is lower triangular

$$\Phi(L)\mathbf{y}_t = \mathbf{c} + \varepsilon_t$$
$$\varepsilon_t = \mathbf{H}^{-1}\mathbf{u}_t$$
$$\Rightarrow \Phi(L)\mathbf{y}_t = \mathbf{c} + \mathbf{H}^{-1}\mathbf{u}_t$$
$$\Rightarrow \mathbf{y}_t = \mu + [\Phi(L)]^{-1}\mathbf{H}^{-1}\mathbf{u}_t$$
$$\mathbf{y}_t = \mu + \mathbf{C}(L)\mathbf{u}_t$$
$$\Rightarrow \mathbf{C}(1) = [\Phi(1)]^{-1}\mathbf{H}^{-1}$$

$$\mathbf{C}(1) = [\Phi(1)]^{-1}\mathbf{H}^{-1}$$
$$\mathbf{C}(1)[\mathbf{C}(1)]' = [\Phi(1)]^{-1}\mathbf{H}^{-1}(\mathbf{H}^{-1})'\{[\Phi(1)]^{-1}\}'$$

$$\mathbf{C}(1)[\mathbf{C}(1)]' = [\Phi(1)]^{-1}\mathbf{\Omega}\{[\Phi(1)]^{-1}\}'$$

Can estimate: $\Phi(1)$ and $\mathbf{\Omega}$ from VAR

Want: Lower triangular matrix $\mathbf{C}(1)$ such that

$$\mathbf{C}(1)[\mathbf{C}(1)]' = [\mathbf{\Phi}(1)]^{-1}\mathbf{\Omega}\{[\mathbf{\Phi}(1)]^{-1}\}'$$

Conclusion: $\mathbf{C}(1)$ is Cholesky factor of

$$[\mathbf{\Phi}(1)]^{-1}\mathbf{\Omega}\{[\mathbf{\Phi}(1)]^{-1}\}'$$

To get $\mathbf{H}$ we then use fact that

$$\mathbf{C}(1) = [\mathbf{\Phi}(1)]^{-1}\mathbf{H}^{-1}$$

$$\mathbf{H} = [\mathbf{C}(1)]^{-1}[\mathbf{\Phi}(1)]^{-1}$$

Summary:

(1) Estimate VAR's by OLS

$$\mathbf{y}_t = \begin{bmatrix} \Delta x_t \\ \Delta n_t \end{bmatrix}$$

$$\mathbf{y}_t = \mathbf{c} + \hat{\boldsymbol{\Phi}}_1 \mathbf{y}_{t-1} + \hat{\boldsymbol{\Phi}}_2 \mathbf{y}_{t-2} + \cdots + \hat{\boldsymbol{\Phi}}_p \mathbf{y}_{t-p} + \hat{\boldsymbol{\varepsilon}}_t$$

$$\hat{\boldsymbol{\Omega}} = T^{-1} \sum_{t=1}^{T} \hat{\boldsymbol{\varepsilon}}_t \hat{\boldsymbol{\varepsilon}}_t'$$

(2) Find Cholesky factor or lower triangular matrix $\widehat{\mathbf{C}}$ such that

$$\widehat{\mathbf{C}}\widehat{\mathbf{C}}' = \hat{\mathbf{Q}}\hat{\boldsymbol{\Omega}}\hat{\mathbf{Q}}'$$

$$\hat{\mathbf{Q}} = (\mathbf{I}_2 - \hat{\boldsymbol{\Phi}}_1 - \hat{\boldsymbol{\Phi}}_2 - \cdots - \hat{\boldsymbol{\Phi}}_p)^{-1}$$

(3) Technology shock and demand shock for date t are first and second elements of

$$\hat{\mathbf{u}}_t = \hat{\mathbf{B}}_0 \hat{\boldsymbol{\varepsilon}}_t$$

where

$$\hat{\mathbf{B}}_0 = \widehat{\mathbf{C}}^{-1} \hat{\mathbf{Q}}$$

(4) Effect of tech shock or demand shock at date t on $\mathbf{y}_{t+s}$ are given by first and second columns, respectively, of

$$\frac{\partial \mathbf{y}_{t+s}}{\partial \mathbf{u}_t'} = \mathbf{\Psi}_s \mathbf{B}_0^{-1}$$

More generally, if $\mathbf{y}_t$ is n-dimensional vector of differences, long-run effect of structural shock j on level of y_i is given by row i, col j of $[\mathbf{\Phi}(1)]^{-1}\mathbf{B}_0^{-1}$.

If this is postulated to be zero for some subset of i and j can use this as set of restrictions on $\mathbf{B}_0$ along with zero or other restrictions to maximize

$(T/2)\log|\mathbf{B}_0|^2 - (T/2)\log|\mathbf{D}|$
$-(T/2)\text{trace}\{\mathbf{B}_0'\mathbf{D}^{-1}\mathbf{B}_0\hat{\mathbf{\Omega}}\}$

Drawbacks:

(1) $\hat{\mathbf{Q}} = (\mathbf{I}_2 - \hat{\mathbf{\Phi}}_1 - \hat{\mathbf{\Phi}}_2 - \cdots - \hat{\mathbf{\Phi}}_p)^{-1}$

is estimated poorly, sensitive to p

(2) technology shock could be temporary (e.g., delay in adoption of discovered technology)

(3) demand shock could be permanent (e.g., lost human capital)

Structural vector autoregressions 2

A. Problem statement
B. Identification using long-run restrictions
C. Identification using high-frequency data

Faust, Swanson, and Wright (JME, 2004)

Observe: some financial variables move dramatically after Fed announces target change

Inference: these changes reflect the effects of policy

Goal: can we somehow use this to identify VAR?

d particular day in sample

$t(d)$ month associated with day d

f_d^h h-month fed funds futures rate on day d

r_t avg. fed funds rate for month t

assumption: $f_d^h = E_d(r_{t(d)+h})$

Implications:

$f_d^0 - f_{d-1}^0 =$ size of shock to fed policy on day d

$$\frac{f_d^h - f_{d-1}^h}{f_d^0 - f_{d-1}^0} = \frac{\partial E_{t(d)} r_{t+h}}{\partial u_{ft}}$$

where u_{ft} is change in fed policy in month t

$$\frac{f_d^h - f_{d-1}^h}{f_d^0 - f_{d-1}^0} = \frac{\partial E_{t(d)} r_{t+h}}{\partial u_{ft}}$$

Average value for all days d on which there is a target change gives estimate of

$$\gamma_h = \frac{\partial E_t r_{t+h}}{\partial u_{ft}}$$

VAR (reduced-form)

$$\mathbf{y}_t = \mathbf{c} + \mathbf{\Phi}_1 \mathbf{y}_{t-1} + \mathbf{\Phi}_2 \mathbf{y}_{t-2} + \cdots + \mathbf{\Phi}_p \mathbf{y}_{t-p} + \boldsymbol{\varepsilon}_t$$

Structural model:

$$\mathbf{B}_0 \mathbf{y}_t = \mathbf{b}_0 + \mathbf{B}_1 \mathbf{y}_{t-1} + \mathbf{B}_2 \mathbf{y}_{t-2} + \cdots + \mathbf{B}_p \mathbf{y}_{t-p} + \mathbf{u}_t$$

Relation:

$$\mathbf{u}_t = \mathbf{B}_0 \boldsymbol{\varepsilon}_t$$

Suppose shock to fed policy is represented by

$$u_{ft} = \mathbf{e}_4' \mathbf{u}_t = \mathbf{e}_4' \mathbf{B}_0 \boldsymbol{\varepsilon}_t$$

Reduced-form MA representation:

$$\mathbf{y}_t = \boldsymbol{\mu} + \boldsymbol{\varepsilon}_t + \boldsymbol{\Psi}_1 \boldsymbol{\varepsilon}_{t-1} + \boldsymbol{\Psi}_2 \boldsymbol{\varepsilon}_{t-2} + \cdots$$

$$\boldsymbol{\varepsilon}_t = \mathbf{y}_t - \hat{\mathbf{y}}_{t|t-1}$$

$$\mathbf{y}_t = \boldsymbol{\mu} + \mathbf{B}_0^{-1}\mathbf{B}_0 \boldsymbol{\varepsilon}_t + \boldsymbol{\Psi}_1 \mathbf{B}_0^{-1}\mathbf{B}_0 \boldsymbol{\varepsilon}_{t-1} + \boldsymbol{\Psi}_2 \mathbf{B}_0^{-1}\mathbf{B}_0 \boldsymbol{\varepsilon}_{t-2} + \cdots$$

$$\mathbf{y}_t = \boldsymbol{\mu} + \mathbf{B}_0^{-1}\mathbf{B}_0 \boldsymbol{\varepsilon}_t + \boldsymbol{\Psi}_1 \mathbf{B}_0^{-1}\mathbf{B}_0 \boldsymbol{\varepsilon}_{t-1} + \boldsymbol{\Psi}_2 \mathbf{B}_0^{-1}\mathbf{B}_0 \boldsymbol{\varepsilon}_{t-2} + \cdots$$

$$\mathbf{y}_t = \boldsymbol{\mu} + \mathbf{B}_0^{-1}\mathbf{u}_t + \boldsymbol{\Psi}_1 \mathbf{B}_0^{-1}\mathbf{u}_{t-1} + \boldsymbol{\Psi}_2 \mathbf{B}_0^{-1}\mathbf{u}_{t-2} + \cdots$$

$$\frac{\partial r_{t+h}}{\partial u_{ft}} = \mathbf{e}_4' \boldsymbol{\Psi}_h \mathbf{b}^{(4)} = \gamma_h$$

where $\mathbf{b}^{(4)}$ is fourth column of $\mathbf{B}_0^{-1}$

$$\frac{\partial r_{t+h}}{\partial u_{ft}} = \mathbf{e}_4' \boldsymbol{\Psi}_h \mathbf{b}^{(4)} = \gamma_h$$

Can estimate:

$\boldsymbol{\Psi}_h$ from estimated monthly VAR

γ_h from daily target change data

$$\hat{\gamma}_h = \text{ average } \frac{f_d^h - f_{d-1}^h}{f_d^0 - f_{d-1}^0}$$

$$\mathbf{e}_4' \boldsymbol{\Psi}_h \mathbf{b}^{(4)} = \gamma_h$$

Let $\boldsymbol{\psi}_{4h}' = \mathbf{e}_4' \boldsymbol{\Psi}_h$

Then:

$$\begin{aligned} \boldsymbol{\psi}_{40}' \mathbf{b}^{(4)} &= \gamma_0 \\ \boldsymbol{\psi}_{41}' \mathbf{b}^{(4)} &= \gamma_1 \\ &\vdots \\ \boldsymbol{\psi}_{4,n-1}' \mathbf{b}^{(4)} &= \gamma_{n-1} \end{aligned}$$

$$\begin{bmatrix} \psi_{40}' \\ \psi_{41}' \\ \vdots \\ \psi_{4,n-1}' \end{bmatrix} \mathbf{b}^{(4)} = \begin{bmatrix} \gamma_0 \\ \gamma_1 \\ \vdots \\ \gamma_{n-1} \end{bmatrix}$$

Let

$$\mathbf{H} = \begin{bmatrix} \psi_{40}' \\ \psi_{41}' \\ \vdots \\ \psi_{4,n-1}' \end{bmatrix} \quad \boldsymbol{\gamma} = \begin{bmatrix} \gamma_0 \\ \gamma_1 \\ \vdots \\ \gamma_{n-1} \end{bmatrix}$$

$$\mathbf{H}\mathbf{b}^{(4)} = \boldsymbol{\gamma}$$

Both $\mathbf{H}$ and $\boldsymbol{\gamma}$ can be estimated.

If rows of $\widehat{\mathbf{H}}$ are linearly independent, then

$$\hat{\mathbf{b}}^{(4)} = \widehat{\mathbf{H}}^{-1} \hat{\boldsymbol{\gamma}}$$

Summary:

We assumed that we can use daily interest rate data to infer effect of policy shock on future interest rates:

$$\frac{\partial r_{t+h}}{\partial u_{ft}}$$

But now we can calculate effect of policy shock on any variable:

$$\frac{\partial \mathbf{y}_{t+s}}{\partial u_{ft}} = \mathbf{\Psi}_s \mathbf{b}^{(4)}$$

Problem: the matrix $\hat{\mathbf{H}}$ does not appear to have full rank.

Solution: Calculate confidence sets under partial identification rather than point estimates

$$\hat{\mathbf{H}} = \begin{bmatrix} \hat{\Psi}'_{40} \\ \hat{\Psi}'_{41} \\ \vdots \\ \hat{\Psi}'_{4,n-1} \end{bmatrix}$$

$$T^{1/2}\left[\text{vec}(\hat{\mathbf{H}} - \mathbf{H}_0)\right] \xrightarrow{L} N(\mathbf{0}, \mathbf{R})$$

R can be consistently estimated from VAR distribution

(e.g., simulate draws from asymptotic distribution of $\{\hat{\Phi}_s\}$ and calculate $\{\hat{\Psi}_s\}$ for each draw)

$$T^{1/2}(\hat{\gamma} - \gamma_0) \xrightarrow{L} N(\mathbf{0}, \mathbf{G})$$

G can be consistently estimated from covariance of futures observations

$H_0: \underset{(g\times n)(n\times 1)}{\mathbf{H}\ \mathbf{b}^{(4)}} = \underset{(g\times 1)}{\boldsymbol{\gamma}}$

If $\mathbf{b}^{(4)}$ is the true value, then

$$S(\mathbf{b}^{(4)}) = T(\widehat{\mathbf{H}}\mathbf{b}^{(4)} - \hat{\boldsymbol{\gamma}})' \times \left[(\mathbf{b}^{(4)\prime} \otimes \mathbf{I}_g)\hat{\mathbf{R}}(\mathbf{b}^{(4)} \otimes \mathbf{I}_g) + \widehat{\mathbf{G}}\right]^{-1} \times(\widehat{\mathbf{H}}\mathbf{b}^{(4)} - \hat{\boldsymbol{\gamma}}) \xrightarrow{p} \chi^2(g)$$

The set A of all values $\mathbf{b}^{(4)}$ such that

$$S(\mathbf{b}^{(4)}) \leq c$$

where c is 95% critical value for $\chi^2(g)$ then is a 95% confidence set for $\mathbf{b}^{(4)}$

For statistic such as structural impulse-response coefficients, use Bonferroni to find outer bounds on 90% confidence interval.

e.g., let h_{is} = row i element of $\mathbf{\Psi}_s \mathbf{b}^{(4)}$

(1) For any given $\mathbf{b}^{(4)}$ use distribution of $\hat{\mathbf{\Psi}}_s$ to find 95% upper and lower bounds $h_{is}^{(u)}(\mathbf{b}^{(4)})$ and $h_{is}^{(l)}(\mathbf{b}^{(4)})$

(2) Find the value $\mathbf{b}^{(u)} \in A$ for which $h_{is}^{(u)}(\mathbf{b}^{(4)})$ is largest and the value $\mathbf{b}^{(l)} \in A$ for which $h_{is}^{(l)}(\mathbf{b}^{(4)})$ is smallest.

(3) 90% confidence interval for h_{is} is $[h_{is}^{(l)}(\mathbf{b}^{(l)}), h_{is}^{(u)}(\mathbf{b}^{(u)})]$

Structural vector autoregressions 2

A. Problem statement
B. Identification using long-run restrictions
C. Identification using high-frequency data
D. Identification using external instruments

Stock and Watson (BPEA, 2012)
Suppose:
(1) structural shocks $u_{1t}, \ldots, u_{nt}$
are mutually uncorrelated
(2) have instrument z_{it} that is relevant
$E(z_{it}u_{it}) = \alpha_i \neq 0$ and valid
$E(z_{it}u_{jt}) = 0$ for $i \neq j$

Under the above assumptions,
$E(\boldsymbol{\varepsilon}_t z_{it}) = \mathbf{B}_0^{-1} E(\mathbf{u}_t z_{it}) = \mathbf{B}_0^{-1} \alpha \mathbf{e}_i$
so can estimate ith column of
$\mathbf{B}_0^{-1}$ (up to unknown constant) by
$\tilde{\mathbf{b}}^{(i)} = T^{-1} \sum_{t=1}^{T} \hat{\boldsymbol{\varepsilon}}_t z_{it}$

Can normalize by defining shock u_{it}
to be something that increases y_{it}
by one unit: $\hat{\mathbf{b}}^{(i)} = \tilde{\mathbf{b}}^{(i)} / \tilde{b}_i^{(i)}$
$\widehat{\frac{\partial \mathbf{y}_{t+s}}{\partial u_{it}}} = \hat{\boldsymbol{\Psi}}_s \hat{\mathbf{b}}^{(i)}$

Can also estimate $\hat{u}_{it}$ as follows. Suppose we observed $\mathbf{u}_t$ and regressed z_{it} on $\mathbf{u}_t$:

$z_{it} = \boldsymbol{\pi}_i'\mathbf{u}_t + v_{it}$

plim $\hat{\boldsymbol{\pi}}_i = (\alpha/d_{ii})\mathbf{e}_i$

If instead we regressed z_{it} on $\boldsymbol{\varepsilon}_t$,

$z_{it} = \boldsymbol{\lambda}_i'\boldsymbol{\varepsilon}_t + v_{it}$

this would just be rotation of above regression since $\boldsymbol{\varepsilon}_t = \mathbf{B}_0^{-1}\mathbf{u}_t$

Hence fitted values from regression of z_{it} on $\hat{\boldsymbol{\varepsilon}}_t$ give consistent estimate of $(\alpha/d_{ii})u_{it}$

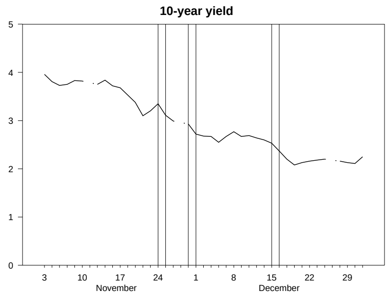
Illustration:

Using high-frequency market response to Fed announcements to identify effects of unconventional monetary policy (Gertler and Karadi, 2013)

Event study methodology

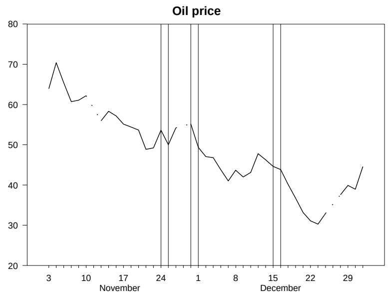
- Nov 25, 2008: LSAP announced
- Dec 1, 2008: Bernanke: “could purchase longer-term Treasury… in substantial quantities”
- Dec 16, 2008: FOMC “stands ready to expand its purchases of agency debt and mortgage-backed securities”
- Mar 18, 2009: Announced new purchases of MBS and agency debt

10-year yield fell 170 bp Nov 3 - Dec 31

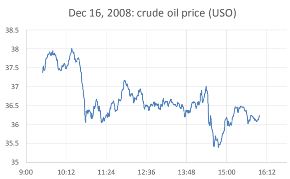


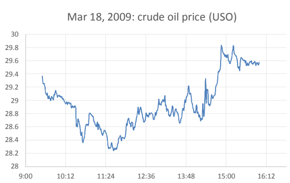
•fell 61 bp on 3 indicated dates

Oil price declined 30% Nov 3 - Dec 31

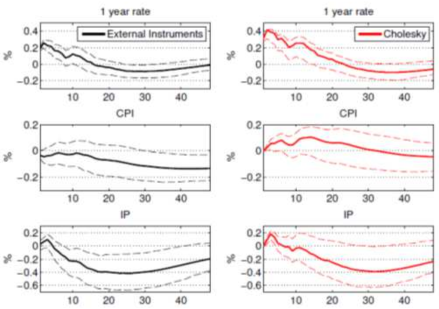


- fell 19% on 3 indicated dates





z_{it} = change in 1-year yield within 30-minute window of key Fed announcemt in month t (= 0 if no event in month t)



Source: Gertler and Karadi (2013)

Structural vector autoregressions 2

A. Problem statement
B. Identification using long-run restrictions
C. Identification using high-frequency data
D. Identification using external instruments
E. Identification using heteroskedasticity

Rigabon and Sack, JME, 2004
Wright, Econ J., 2012
Suppose $\mathbf{y}_t$ consists of high-frequency observations (e.g., daily changes in interest rates, exchange rates, stock prices, commodity prices)

$\mathbf{y}_t = \mathbf{c} + \mathbf{\Phi}_1 \mathbf{y}_{t-1} + \cdots + \mathbf{\Phi}_p \mathbf{y}_{t-p} + \boldsymbol{\varepsilon}_t$
$\boldsymbol{\varepsilon}_t = \mathbf{B}_0^{-1} \mathbf{u}_t$
u_{1t} = monetary policy shock
want to estimate $\mathbf{b}^{(1)}$ (first column of $\mathbf{B}_0^{-1}$)

Suppose we believed that:
(1) monetary policy shocks have higher variance on particular days

$$E(u_{1t}^2) = \begin{cases} d_{11}^{(0)} + \lambda & \text{if } t \in S \\ d_{11}^{(0)} & \text{if } t \notin S \end{cases}$$

Set S is known (e.g., FOMC dates)

(2) A monetary policy shock of given size would have the same effects on these dates as others
(3) Variance and effects of other shocks same on these dates as others

Then

$$E(\mathbf{u}_t\mathbf{u}_t') = \begin{cases} \mathbf{D} + \lambda \mathbf{e}_1\mathbf{e}_1' & \text{if } t \in S \\ \mathbf{D} & \text{if } t \notin S \end{cases}$$

$\mathbf{e}_1 =$ col 1 of $\mathbf{I}_n$

$$\boldsymbol{\varepsilon}_t = \mathbf{B}_0^{-1}\mathbf{u}_t = \sum_{i=1}^n \mathbf{b}^{(i)} u_{it}$$

$$E(\boldsymbol{\varepsilon}_t\boldsymbol{\varepsilon}_t') = \begin{cases} \mathbf{B}_0^{-1}\mathbf{D}(\mathbf{B}_0^{-1})' + \lambda \mathbf{b}^{(1)}(\mathbf{b}^{(1)})' & \text{if } t \in S \\ \mathbf{B}_0^{-1}\mathbf{D}(\mathbf{B}_0^{-1})' & \text{if } t \notin S \end{cases}$$

$\hat{\mathbf{\Omega}}_1 = T_1^{-1} \sum_{t=1}^{T} \hat{\boldsymbol{\varepsilon}}_t \hat{\boldsymbol{\varepsilon}}_t' \delta(t \in S)$
$T_1 = \sum_{t=1}^{T} \delta(t \in S)$
$\hat{\mathbf{\Omega}}_0 = T_0^{-1} \sum_{t=1}^{T} \hat{\boldsymbol{\varepsilon}}_t \hat{\boldsymbol{\varepsilon}}_t' \delta(t \notin S)$
$T_0 = \sum_{t=1}^{T} \delta(t \notin S)$
$\hat{\mathbf{\Omega}}_1 - \hat{\mathbf{\Omega}}_0 \xrightarrow{p} \lambda \mathbf{b}^{(1)} (\mathbf{b}^{(1)})'$
so we can estimate $\mathbf{b}^{(1)}$ up to an unknown scale, e.g.: normalize $\lambda = 1$

$\sqrt{T_1}\,[\text{vech}(\hat{\mathbf{\Omega}}_1) - \text{vech}(\mathbf{\Omega}_1)]$
$\xrightarrow{L} N(\mathbf{0}, \mathbf{V}_1)$
element of $\mathbf{V}_1$ corresponding to covariance between $\hat{\sigma}_{ij}$ and $\hat{\sigma}_{\ell m}$ given by $(\sigma_{i\ell}\sigma_{jm} + \sigma_{im}\sigma_{j\ell})$
(Hamilton, TSA, p. 301).

(1) Test null hypothesis that $\mathbf{\Omega}_0 = \mathbf{\Omega}_1$
$\hat{\mathbf{q}}'[\hat{\mathbf{V}}_1/T_1 + \hat{\mathbf{V}}_0/T_0]^{-1}\hat{\mathbf{q}} \xrightarrow{L} \chi^2(n(n+1)/2)$
$\hat{\mathbf{q}} =$ vech($\hat{\mathbf{\Omega}}_1$) –vech($\hat{\mathbf{\Omega}}_0$)
or bootstrap critical value
(should reject H_0 if assumptions correct)

(2) Estimate $\mathbf{b}^{(1)}$ by minimum chi square:

$$\hat{\mathbf{b}}^{(1)} = \underset{\mathbf{b}^{(1)}}{\arg\min}\ \tilde{\mathbf{q}}'[\hat{\mathbf{V}}_1/T_1 + \hat{\mathbf{V}}_0/T_0]^{-1}\tilde{\mathbf{q}}$$

$$\tilde{\mathbf{q}} = \hat{\mathbf{q}} - \text{vech}[\mathbf{b}^{(1)}(\mathbf{b}^{(1)})']$$

$$\widehat{\frac{\partial \mathbf{y}_{t+s}}{\partial u_{1t}}} = \hat{\boldsymbol{\Psi}}_s\hat{\mathbf{b}}^{(1)}$$

(3) Test null hypothesis restriction valid: value of objective function asymptotically $\chi^2(n(n-1)/2)$ or bootstrap critical value (should not reject H_0 if assumptions correct)

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C. Identification using high-frequency data
D. Identification using external instruments
E. Identification using heteroskedasticity
F. Identification using sign restrictions

Rubio-Ramirez, Waggoner, and Zha, Rev Econ Studies, 2010.
We can achieve partial identification with sign restrictions such as: monetary policy shock raises short-term rate and lowers output and inflation

Even if true $\mathbf{\Omega}$ is known, we could only infer that $\mathbf{B}_0^{-1} \in S(\mathbf{\Omega})$.
$\Rightarrow$ $\mathbf{B}_0$ is set-identified, not point-identified

Baumeister and Hamilton, "Sign Restrictions, Structural Vector Autoregressions, and Useful Prior Information", Econometrics seminar, Thursday Nov 21