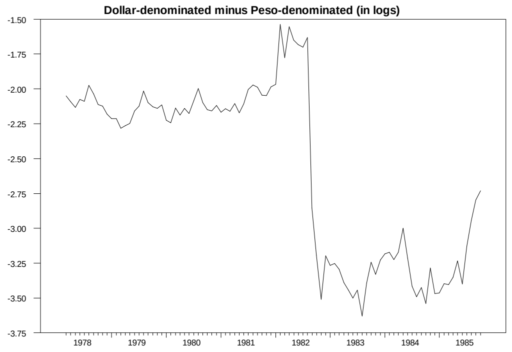
Markov-switching processes



Model of structural change:

$$y_t - \mu_1 = \phi(y_{t-1} - \mu_1) + \varepsilon_t \qquad t \leq t_0$$
$$y_t - \mu_2 = \phi(y_{t-1} - \mu_2) + \varepsilon_t \qquad t > t_0$$

Questions:

1) How forecast with this model?
2) What caused change at t_0?
3) What is probability law for $\{y_t\}$?

$$s_t^* = 1 \quad t = 1, 2, \ldots, t_0$$
$$s_t^* = 2 \quad t = t_0 + 1, t_0 + 2, \ldots$$
$$y_t - \mu_{s_t^*} = \phi(y_{t-1} - \mu_{s_{t-1}^*}) + \varepsilon_t$$

Need: probability law for s_t^*

Markov chain:

$$P(s_t^* = j | s_{t-1}^* = i, s_{t-2}^* = k, \ldots)$$
$$= P(s_t^* = j | s_{t-1}^* = i)$$
$$= p_{ij}$$

Transition from 1 to 2 is permanent

$$\Rightarrow p_{21} = 0$$

In general, if s_t is a Markov chain taking on one of the values $s_t = 1, 2, \ldots, N$, let $p_{ij} = P(s_t = j | s_{t-1} = i)$. Collect in matrix **P** $= [p_{ji}]$

$$\mathbf{P} = \begin{bmatrix} p_{11} & p_{21} & \cdots & p_{N1} \\ p_{12} & p_{22} & \cdots & p_{N2} \\ \vdots & \vdots & \cdots & \vdots \\ p_{1N} & p_{2N} & \cdots & p_{NN} \end{bmatrix}$$

Let $\xi_t = \mathbf{e}_i$ (the ith column of $\mathbf{I}_N$) when $s_t = i$. Then

$$E(\xi_{t+1} | \xi_t = \mathbf{e}_i) = \begin{bmatrix} P(s_{t+1} = 1 | s_t = i) \\ P(s_{t+1} = 2 | s_t = i) \\ \vdots \\ P(s_{t+1} = N | s_t = i) \end{bmatrix}$$

$$= \mathbf{P}\mathbf{e}_i$$

$$= \mathbf{P}\xi_t$$

$$\xi_{t+1} = \mathbf{P}\xi_t + \mathbf{v}_{t+1} \qquad \mathbf{v}_{t+1} \sim m.d.s.$$

More generally, suppose we had a set of observations $\Omega_t = \{\mathbf{y}_t, \mathbf{y}_{t-1}, \ldots, \mathbf{y}_1\}$ that gave us an imperfect inference about s_t summarized as

$$\hat{\xi}_{t|t} = E(\xi_t | \Omega_t) = \begin{bmatrix} P(s_t = 1 | \Omega_t) \\ P(s_t = 2 | \Omega_t) \\ \vdots \\ P(s_t = N | \Omega_t) \end{bmatrix}$$

Then

$$\hat{\xi}_{t+1|t} = E(\xi_{t+1}|\Omega_t) = \mathbf{P}\hat{\xi}_{t|t}$$

(e.g., row j states that

$$\begin{aligned} P(s_{t+1} = j|\Omega_t) \\ &= p_{1j}P(s_t = 1|\Omega_t) + p_{2j}P(s_t = 2|\Omega_t) \\ &\quad + \cdots + p_{Nj}P(s_t = N|\Omega_t) \end{aligned}$$

Return to original example of interest:

$$y_t - \mu_{s_t^*} = \phi(y_{t-1} - \mu_{s_{t-1}^*}) + \varepsilon_t$$

$\varepsilon_t \sim$ i.i.d. $N(0,\sigma^2)$

$$P(s_t^* = j|s_{t-1}^* = i) = p_{ij}^* \quad i,j = 1,2$$

$\{s_t^*\}_{t=1}^T$ independent of $\{\varepsilon_t\}_{t=1}^T$

$$\Omega_t = \{y_t, y_{t-1}, \ldots, y_1\}$$

Implication:

$$y_t|\Omega_{t-1}, s_t^*, s_{t-1}^* \sim N\left(\mu_{s_t^*} + \phi(y_{t-1} - \mu_{s_{t-1}^*}), \sigma^2\right)$$

Convenient to summarize $\{s_t^*, s_{t-1}^*\}$ with a single Markov chain:

$s_t = 1$ if $s_t^* = 1$ and $s_{t-1}^* = 1$

$s_t = 2$ if $s_t^* = 2$ and $s_{t-1}^* = 1$

$s_t = 3$ if $s_t^* = 1$ and $s_{t-1}^* = 2$

$s_t = 4$ if $s_t^* = 2$ and $s_{t-1}^* = 2$

$\xi_t = \mathbf{e}_i$ (ith column of $\mathbf{I}_4$) when $s_t = i$

$$\hat{\xi}_{t+1|t} = \mathbf{P}\hat{\xi}_{t|t}$$

$$\mathbf{P} = \begin{bmatrix} p_{11}^* & 0 & p_{11}^* & 0 \\ p_{12}^* & 0 & p_{12}^* & 0 \\ 0 & p_{21}^* & 0 & p_{21}^* \\ 0 & p_{22}^* & 0 & p_{22}^* \end{bmatrix}$$

$$\begin{aligned} p(y_t | s_t = 3, \Omega_{t-1}) \\ &= p(y_t | \mu_{s_t^*} = \mu_1, \mu_{s_{t-1}^*} = \mu_2, \Omega_{t-1}) \\ &= \frac{1}{\sqrt{2\pi}\,\sigma} \exp\left\{ \frac{-[y_t - \mu_1 - \phi(y_{t-1} - \mu_2)]^2}{2\sigma^2} \right\} \end{aligned}$$

Collect the densities that might be associated with each of the $N = 4$ states in an $(N \times 1)$ vector

$$\boldsymbol{\eta}_t = \begin{bmatrix} p(y_t|s_t = 1, \Omega_{t-1}) \\ p(y_t|s_t = 2, \Omega_{t-1}) \\ \vdots \\ p(y_t|s_t = N, \Omega_{t-1}) \end{bmatrix}$$

Recall that

$$\mathbf{P}\hat{\boldsymbol{\xi}}_{t-1|t-1} = \begin{bmatrix} P(s_t = 1|\Omega_{t-1}) \\ P(s_t = 2|\Omega_{t-1}) \\ \vdots \\ P(s_t = N|\Omega_{t-1}) \end{bmatrix}$$

Thus

$$\boldsymbol{\eta}_t \odot \mathbf{P}\hat{\boldsymbol{\xi}}_{t-1|t-1} =$$

$$\begin{bmatrix} p(y_t|s_t = 1, \Omega_{t-1})P(s_t = 1|\Omega_{t-1}) \\ p(y_t|s_t = 2, \Omega_{t-1})P(s_t = 2|\Omega_{t-1}) \\ \vdots \\ p(y_t|s_t = N, \Omega_{t-1})P(s_t = N|\Omega_{t-1}) \end{bmatrix}$$

Summing the elements of this vector gives

$$\mathbf{1}'(\eta_t \odot \mathbf{P}\hat{\xi}_{t-1|t-1})$$
$$= \sum_{j=1}^{N} p(y_t, s_t = j|\Omega_{t-1})$$
$$= p(y_t|\Omega_{t-1}),$$

the conditional likelihood of tth observation.

The result of dividing the jth element of $(\eta_t \odot \mathbf{P}\hat{\xi}_{t-1|t-1})$ by the conditional likelihood is

$$\frac{p(y_t, s_t = j|\Omega_{t-1})}{p(y_t|\Omega_{t-1})} = P(s_t = j|y_t, \Omega_{t-1})$$

$$\frac{(\eta_t \odot \mathbf{P}\hat{\xi}_{t-1|t-1})}{\mathbf{1}'(\eta_t \odot \mathbf{P}\hat{\xi}_{t-1|t-1})} = \hat{\xi}_{t|t}$$

$$\frac{(\eta_t \odot \mathbf{P}\hat{\xi}_{t-1|t-1})}{\mathbf{1}'(\eta_t \odot \mathbf{P}\hat{\xi}_{t-1|t-1})} = \hat{\xi}_{t|t}$$

Iterative algorithm similar to Kalman filter:

Input for step t:

$\hat{\xi}_{t-1|t-1}$

(an $N \times 1$ vector whose jth element is $P(s_{t-1} = j|y_{t-1}, y_{t-2}, \ldots, y_1)$).

Output for step t:

$\hat{\xi}_{t|t}$

Options for initial value $\hat{\xi}_{0|0}$:

(1) If Markov chain is ergodic, use ergodic probabilities

$$\hat{\xi}_{0|0} = (\mathbf{A}'\mathbf{A})^{-1}\mathbf{A}'\mathbf{e}_{N+1}$$

$$\underset{(N+1)\times N}{\mathbf{A}} = \begin{bmatrix} \mathbf{I}_N - \mathbf{P} \\ \mathbf{1}' \end{bmatrix}$$

(2) Set $\hat{\xi}_{0|0} = \boldsymbol{\rho}$, a vector of free parameters to be estimated by maximum likelihood or Bayesian methods along with the other parameters.

(3) Set $\hat{\xi}_{0|0} = N^{-1}\mathbf{1}$.

(4) Set $\hat{\xi}_{0|0}$ based on prior beliefs.

Above assumed we knew parameters $\boldsymbol{\theta}$ appearing in $\boldsymbol{\eta}_t = [p(y_t|s_t = j, \Omega_{t-1}; \boldsymbol{\theta})]_{j=1}^{N}$ (in this case $\boldsymbol{\theta} = (\phi, \mu_1, \mu_2, \sigma^2)'$) and $\mathbf{p}$ appearing in $\mathbf{P}$ (in this case $\mathbf{p} = (p_{11}, p_{22})'$).

However, as byproduct of step t of iteration we ended up calculating $p(y_t|\Omega_{t-1}; \boldsymbol{\theta}, \mathbf{p})$ and so we've calculated log likelihood

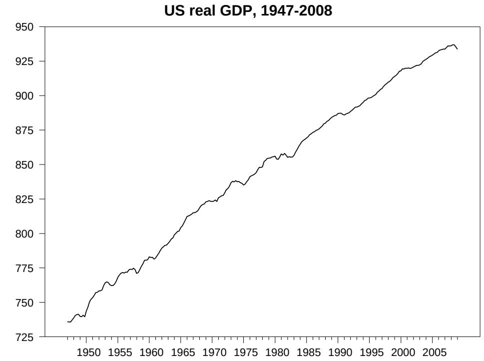
$$\mathcal{L}(\boldsymbol{\theta}, \mathbf{p}) = \sum_{t=1}^{T} \log p(y_t|\Omega_{t-1}; \boldsymbol{\theta}, \mathbf{p})$$

which can be maximized numerically with respect to $\boldsymbol{\theta}$ and $\mathbf{p}$ by numerical methods.

Note– during numerical search we'd want to be choosing λ_{11} and λ_{22} rather than p_{11} and p_{22} where

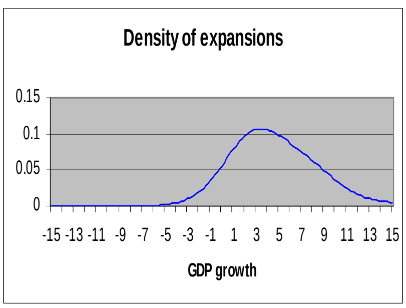
$$p_{11} = \frac{\lambda_{11}^2}{1+\lambda_{11}^2}$$

$$p_{22} = \frac{\lambda_{22}^2}{1+\lambda_{22}^2}$$

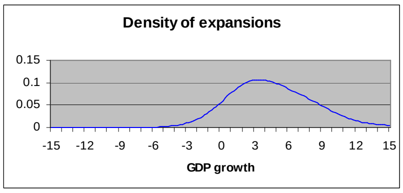


Measured as 100 times natural log
(means vertical change of 10 is approximately a 10% increase)

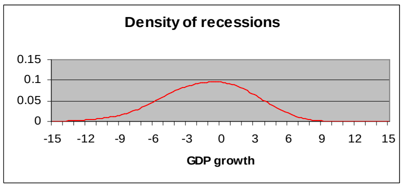
cumulative drop, 2007:Q3 to 2009:Q1 = 2.4%



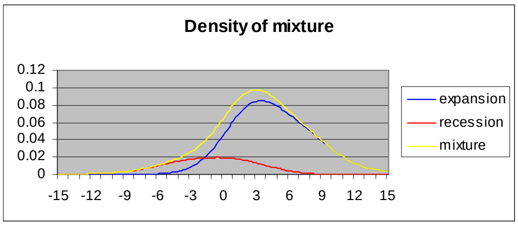
μ = 4.7
σ = 3.5

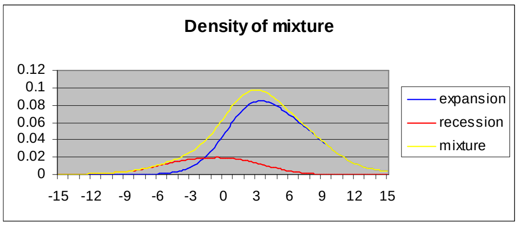


$\mu = 4.7$
$\sigma = 3.5$



$\mu = -1.2$
$\sigma = 3.5$

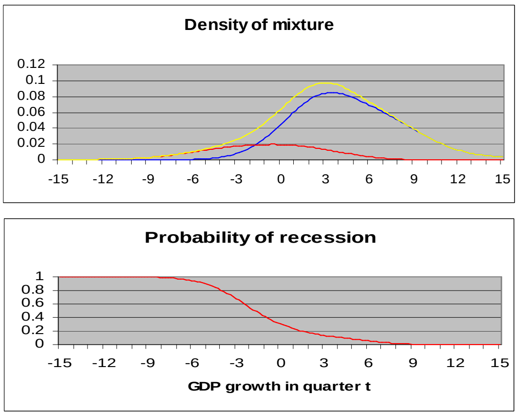




$$\Pr(S_t = 2 \mid y_t) = \frac{\Pr(S_t = 2, y_t)}{f(y_t)}$$

$$= \frac{\Pr(S_t = 2, y_t)}{\Pr(S_t = 1, y_t) + \Pr(S_t = 2, y_t)}$$

$$\Pr(S_t = 2, y_t) = \Pr(S_t = 2) \cdot f(y_t \mid S_t = 2)$$



Prob(NBER expansion in t + 1| NBER expansion in t)
= 164/174 = 0.94
Prob(NBER contraction in t +1 | NBER contraction in t)
= 35/44 = 0.80

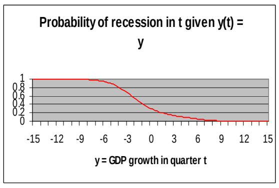
$$\Pr(S_{t+1}=2 \mid y_{t+1}, y_t) = \frac{\Pr(S_{t+1}=2, y_{t+1} \mid y_t)}{f(y_{t+1} \mid y_t)}$$

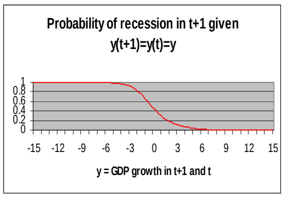
$$= \frac{\Pr(S_{t+1}=2, y_{t+1} \mid y_t)}{\Pr(S_{t+1}=1, y_{t+1} \mid y_t) + \Pr(S_{t+1}=2, y_{t+1} \mid y_t)}$$

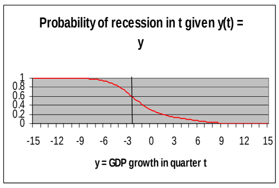
$$\Pr(S_{t+1}=2, y_{t+1} \mid y_t) = \sum_{s=1}^{2} \Pr(S_{t+1}=2, S_t = s, y_{t+1} \mid y_t)$$

$$\Pr(S_{t+1}=2, S_t = s, y_{t+1} \mid y_t) =$$

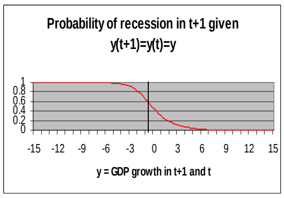
$$\underbrace{\Pr(S_t = s \mid y_t)}_{\substack{\text{known from}\\ \text{previous slide}}} \cdot \underbrace{\Pr(S_{t+1}=2 \mid S_t = s, y_t)}_{\substack{0.80 \text{ for } s=2\\ 0.06 \text{ for } s=1}} \cdot \underbrace{f(y_{t+1} \mid S_{t+1}=2, S_t = s, y_t)}_{\substack{\text{known from}\\ \text{density of recessions}}}$$







Prob = 0.6
at y = -2.2



Prob = 0.6
At y = -0.8

Can also calculate “smoothed probability”

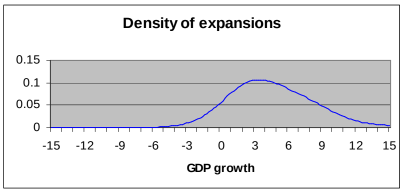
$$\Pr(S_t = 2 \mid y_t, y_{t+1})$$

Nonparametric summary of NBER labels:

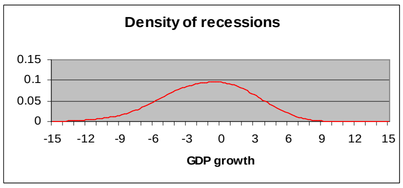
$f(y_t \mid S_t = 1)$ estimated nonparametrically

$\Pr(S_{t+1} = 1 \mid S_t = 1) = 0.94$

$\Pr(S_{t+1} = 2 \mid S_t = 2) = 0.80$



$\mu = 4.7$
$\sigma = 3.5$



$\mu = -1.2$
$\sigma = 3.5$

Could also approach parametrically:

$f(y_t \mid S_t = 1) \sim N(\mu_1, \sigma^2)$

$f(y_t \mid S_t = 2) \sim N(\mu_2, \sigma^2)$

$\Pr(S_{t+1} = 1 \mid S_t = 1) = p_{11}$

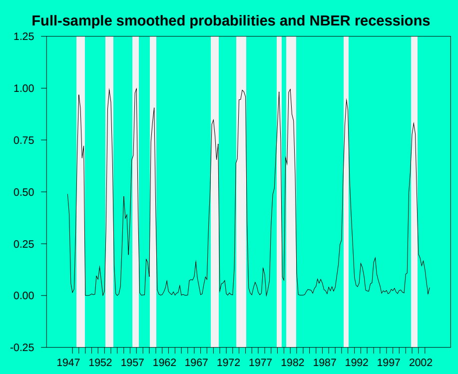
$\Pr(S_{t+1} = 2 \mid S_t = 2) = p_{22}$

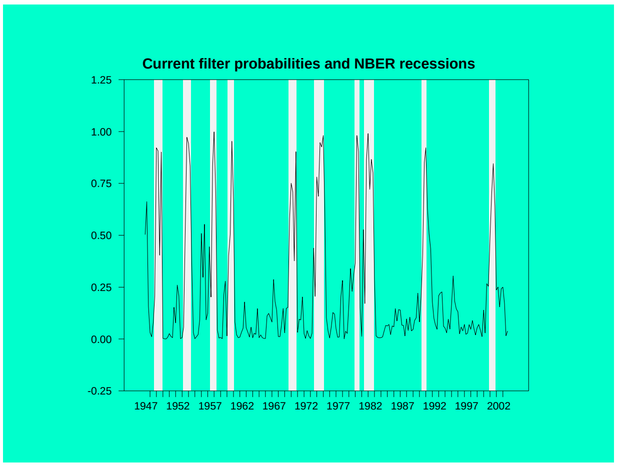
$$\Pr(S_{t+1}=2, S_t=s, y_{t+1} \mid y_t) =$$

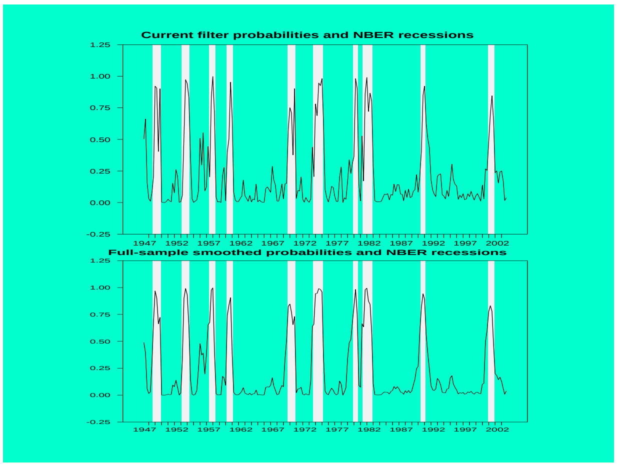
$$\underbrace{\Pr(S_t=s \mid y_t)}_{\substack{\text{known from}\\ \text{previous step}}} \cdot \underbrace{\Pr(S_{t+1}=2 \mid S_t=s, y_t)}_{\substack{\text{p22 for s=2}\\ \text{p12 for s=1}}} \cdot \underbrace{f(y_{t+1} \mid S_{t+1}=2, S_t=s, y_t)}_{N(\mu_2,\sigma^2)}$$

$$\begin{aligned}
&f(y_{t+1} \mid y_t) = \\
&\Pr(S_{t+1}=1, S_t=1, y_{t+1} \mid y_t) + \\
&\Pr(S_{t+1}=1, S_t=2, y_{t+1} \mid y_t) + \\
&\Pr(S_{t+1}=2, S_t=1, y_{t+1} \mid y_t) + \\
&\Pr(S_{t+1}=2, S_t=2, y_{t+1} \mid y_t)
\end{aligned}$$

Parameter	Estimated from NBER dates and GDP	Estimated from GDP alone
μ_1	4.73	4.62
μ_2	-1.19	-0.57
σ	3.47	3.35
p_{11}	0.94	0.92
p_{22}	0.80	0.74



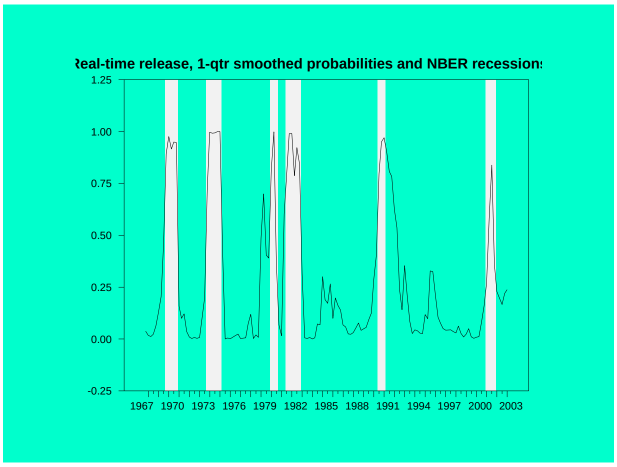


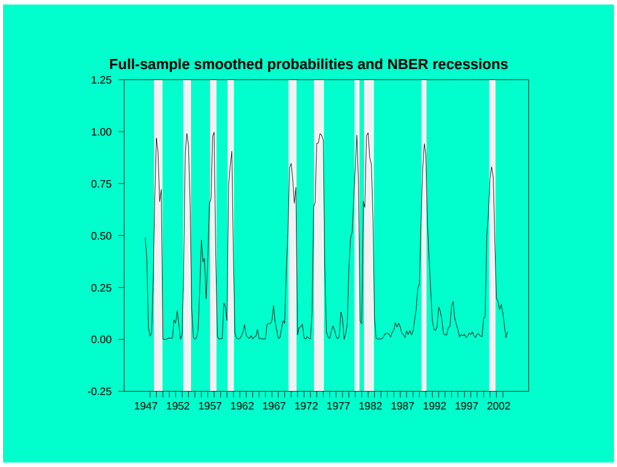


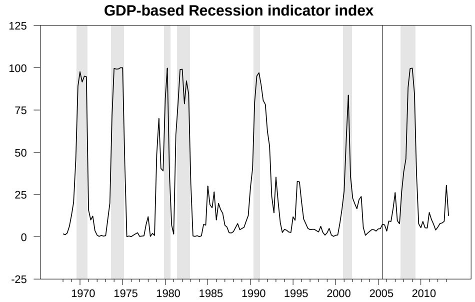
Another challenge:

(1) Use only unrevised data as it was actually available at the time.

(2) Use only data up through quarter t to estimate parameters and form inference for quarter t – 1.


Real-time release, 1-qtr smoothed probabilities and NBER recessions
1.25
1.00
0.75
0.50
0.25
0.00
-0.25
1967 1970 1973 1976 1979 1982 1985 1988 1991 1994 1997 2000 2003


Full-sample smoothed probabilities and NBER recessions
1.25
1.00
0.75
0.50
0.25
0.00
-0.25
1947 1952 1957 1962 1967 1972 1977 1982 1987 1992 1997 2002


GDP-based Recession indicator index
125
100
75
50
25
0
-25
1970 1975 1980 1985 1990 1995 2000 2005 2010

Date of announcement	Announcement
Simulated (through June 2005)	
May 1970	recession began 1969:Q2
Aug 1971	recession ended 1970:Q4
May 1974	recession began 1973:Q4
Feb 1976	recession ended 1975:Q1
Nov 1979	recession began 1979:Q2
May 1981	recession ended 1980:Q2
Feb 1982	recession began 1981:Q2
Aug 1983	recession ended 1982:Q4
Feb 1991	recession began 1989:Q4
Feb 1993	recession ended 1991:Q4
Feb 2002	recession began 2001:Q1
Aug 2002	recession ended 2001:Q3
Actual real time (since July 2005)	
Jan 30, 2009	recession began 2007:Q4
Apr 30, 2010	recession ended 2009:Q2

General case:

$$\boldsymbol{\eta}_t = \begin{bmatrix} p(\mathbf{y}_t|s_t = 1, \Omega_{t-1}) \\ p(\mathbf{y}_t|s_t = 2, \Omega_{t-1}) \\ \vdots \\ p(\mathbf{y}_t|s_t = N, \Omega_{t-1}) \end{bmatrix}$$

$$p(\mathbf{y}_t|\Omega_{t-1}) = \mathbf{1}'(\boldsymbol{\eta}_t \odot \mathbf{P}\hat{\boldsymbol{\xi}}_{t-1|t-1})$$

$$\mathcal{L}(\mathbf{y}_1, \ldots, \mathbf{y}_T; \boldsymbol{\theta}) = \sum_{t=1}^{T} \log p(\mathbf{y}_t|\Omega_{t-1})$$

Example: Dueker (JBES, 1997). AR(1) with Student t innovations whose degrees of freedom change with regime:

$$p(y_t|s_t = j, \Omega_{t-1}; \boldsymbol{\theta}) = \frac{\Gamma[(\nu_j + 1)/2]}{(\pi \nu_j)^{1/2}\Gamma(\nu_j/2)\sigma} \left[1 + \frac{(y_t - c - \phi y_{t-1})^2}{\sigma \nu_j}\right]^{-(\nu_j+1)/2}$$

$$\boldsymbol{\theta} = (c, \phi, \sigma, \nu_1, \nu_2)'$$

Example: Krolzig (*Markov-Switching Vector Autoregressions,* Springer 1997): Gaussian VAR(1) with lag coefficients changing:

$$p(\mathbf{y}_t|s_t = j, \Omega_{t-1}, \boldsymbol{\theta}) = (2\pi)^{-n/2}|\boldsymbol{\Omega}|^{-1/2}$$
$$\exp[-(1/2)(\mathbf{y}_t - \mathbf{c}_j - \boldsymbol{\Phi}_j\mathbf{y}_{t-1})'\boldsymbol{\Omega}^{-1}$$
$$(\mathbf{y}_t - \mathbf{c}_j - \boldsymbol{\Phi}_j\mathbf{y}_{t-1})]$$
$$\boldsymbol{\theta} = (\mathbf{c}_1', \mathbf{c}_2', (\mathrm{vec}\,\Phi_1')', (\mathrm{vec}\,\Phi_2')', (\mathrm{vech}\,\boldsymbol{\Omega})')'$$

Can also allow transition probabilities to be parametric function of exogenous or lagged dependent variables $\mathbf{z}_t$:

$$\mathbf{P} = [P(s_t = j|s_{t-1} = i, \mathbf{z}_t; \mathbf{p}]_{i,j=1}^{N}.$$

Example:

$$\mathbf{P} = \begin{bmatrix} \frac{\exp(\boldsymbol{\gamma}'\mathbf{z}_t)}{1+\exp(\boldsymbol{\gamma}'\mathbf{z}_t)} & \frac{1}{1+\exp(\boldsymbol{\delta}'\mathbf{z}_t)} \\ \frac{1}{1+\exp(\boldsymbol{\gamma}'\mathbf{z}_t)} & \frac{\exp(\boldsymbol{\delta}'\mathbf{z}_t)}{1+\exp(\boldsymbol{\delta}'\mathbf{z}_t)} \end{bmatrix}$$

$$\mathbf{p} = (\boldsymbol{\gamma}', \boldsymbol{\delta}')'$$

Testing the null hypothesis of no change in regime and choosing number of states N.

Example: Markov-switching regression

$$y_t = \mathbf{x}_t'\boldsymbol{\beta}_{s_t} + \sigma_{s_t}\varepsilon_t$$
$$s_t \in \{1, \ldots, N\}$$
$$P(s_t = j|s_{t-1} = i) = p_{ij}$$

H_0: $\boldsymbol{\beta}_j = \boldsymbol{\beta}$, $\sigma_j = \sigma$ for $j = 1, \ldots, N$

Problem: p_{ij} unidentified under H_0.
Usual regularity conditions not satisfied, distribution of likelihood ratio test nonstandard.
(1) Smith, Naik, and Tsai (J. Econometrics, 2006): generalization of Akaike Information Criteria.

$$\begin{aligned}
\boldsymbol{\theta} &= (\boldsymbol{\beta}_1', \ldots, \boldsymbol{\beta}_N', \sigma_1, \ldots, \sigma_N, \\
&\quad p_{ij, i=1,\ldots,N; j=1,..,N-1})' \\
\mathcal{L}(\boldsymbol{\theta}) &= \textstyle\sum_{t=1}^{T} \log p(y_t | \Omega_{t-1}; \boldsymbol{\theta}) \\
\Omega_{t-1} &= \{\mathbf{x}_t, y_{t-1}, \mathbf{x}_{t-1}, \ldots, y_0, \mathbf{x}_0\} \\
\hat{\boldsymbol{\theta}}_{MLE} &= \arg\max \mathcal{L}(\boldsymbol{\theta}) \\
\hat{T}_i &= \textstyle\sum_{t=1}^{T} P(s_t = i | \Omega_T; \hat{\boldsymbol{\theta}}_{MLE})
\end{aligned}$$

Smith, Naik, and Tsai propose choosing model that minimizes

$$MSC = -2\mathcal{L}(\hat{\boldsymbol{\theta}}_{MLE}) + \textstyle\sum_{i=1}^{N} \frac{\hat{T}_i(\hat{T}_i + Nk)}{\hat{T}_i - Nk - 2}$$

$$\boldsymbol{\beta}_i = (k \times 1)$$

MSC is estimate of information loss (Kullback-Leibler divergence) between $p(y_t|\Omega_{t-1};\boldsymbol{\theta}^*)$ and true density $g(y_t|\Omega_{t-1})$ for $\boldsymbol{\theta}^*$ the KL-minimizing value of $\boldsymbol{\theta}$.

(2) Carrasco, Hu, and Ploberger (2013): test of null hypothesis of no switching for more general models.
Let $\ell_t = \log f(y_t|\Omega_{t-1};\boldsymbol{\lambda})$ be the log likelihood under null hypothesis of no switching (e.g., $\boldsymbol{\lambda} = (\boldsymbol{\beta}', \sigma)'$ for previous example)

$\mathbf{h}_t = \frac{\partial \ell_t}{\partial \boldsymbol{\lambda}}\Big|_{\boldsymbol{\lambda}=\hat{\boldsymbol{\lambda}}_{MLE}}$

H_A: first element of $\boldsymbol{\lambda}$ switches from latent Markov chain (CHP also propose tests for vector switches).

$\ell_t^{(1)} =$ first element of $\mathbf{h}_t$

$\ell_t^{(2)} = \frac{\partial^2 \ell_t}{\partial \lambda_1^2}\Big|_{\boldsymbol{\lambda}=\hat{\boldsymbol{\lambda}}_{MLE}}$

$\gamma_t(\rho) = \ell_t^{(2)} + \left[\ell_t^{(1)}\right]^2 + 2\sum_{s<t} \rho^{t-s}\ell_t^{(1)}\ell_s^{(1)}$

$\rho =$ unknown parameter characterizing persistence of Markov chain

(1) Regress $(1/2)\gamma_t(\rho)$ on $\mathbf{h}_t$ and calculate residuals $\hat{\varepsilon}_t(\rho)$.

(2) Calculate

$$C(\rho) = \frac{1}{2}\left[\max\left\{0, \frac{\sum_{t=1}^{T} \gamma_t(\rho)}{2\sqrt{\sum_{t=1}^{T} [\hat{\varepsilon}_t(\rho)]^2}}\right\}\right]^2$$

(3) Find ρ^* over some range (e.g., $\rho \in [0.2, 0.8]$) to maximize $C(\rho)$ by numerical search.

(4) Bootstrap to see if $C(\rho^*)$ is statistically sig (i.e., generate data with no switching and $\boldsymbol{\lambda} = \hat{\boldsymbol{\lambda}}_{MLE}$, calculate $C(\rho^*)$ on each sample).