Supply, Demand, and Specialized Production*

James D. Hamilton
jhamilton@ucsd.edu
University of California at San Diego

March 22, 2021
Revised: February 13, 2023

Abstract

This paper develops a unified model of economic fluctuations and growth characterized by long-run equilibrium unemployment and sustained monopoly power. Changes in demand are a key cause of deviations from the steady-state growth path despite the absence of any nominal rigidities. The key friction in the model is the technological requirement that production of certain goods requires a dedicated team of workers that takes time to assemble and train.

*I am grateful to Douglas Laxton and Johannes Wieland for helpful comments on earlier drafts of this paper.

1 Introduction.

Many of us are persuaded that fluctuations in demand are a key driver of business cycles. Production of automobiles and construction of homes appear to fall in a recession not because the items become more difficult to build, but instead because fewer people seem willing to buy them. Evidence supporting this conclusion comes from Mian and Sufi (2014), Michaillat and Saez (2015), and Auerbach, Gorodnichenko, and Murphy (2020), among many others.

A common understanding of the mechanism whereby a decrease in demand leads to lower output is based on a failure of wages and prices to adjust sufficiently quickly. Potential GDP is sometimes defined as the level of output that would be observed if wages and prices were perfectly flexible. This magnitude is often viewed as depending on the labor force, capital stock, and available technology. If wages and prices fall sufficiently quickly in response to a drop in demand, this is supposed to keep output at potential.

But the key feature that makes developed economies productive is specialization of labor, capital, and technology. When this is the case, potential GDP depends not just on the levels of these factors of production but also on the match between specialized factors and the composition of demand. If the demand for a particular product falls below the level that resources were precommitted to be able to produce, producers have limited incentive to lower price and limited ability to shift productive factors to some other specialization. This paper illustrates this in the context of a general equilibrium growth model with perfectly flexible wages and prices. It develops a unified model of growth and fluctuations in which demand and other variables contribute to short-run fluctuations while long-run growth is determined solely by increases in population and productivity. It thus provides an alternative motivation for the consensus interpretation of economic growth and recessions described above without making any appeal to failure of wages and prices to adjust.

The observation that excess capacity can give rise to real effects of demand shocks even with perfectly flexible wages and prices has recently been developed by Murphy (2017) and Auerbach, Gorodnichenko, and Murphy (2020). Their models share many important features and implications with the one presented here. A key difference is that in their models, capacity is taken to be exogenous, whereas in this paper capacity is determined endogenously as individuals weigh the costs and benefits of specialization.

The model here focuses on the simple case where labor is the only factor of production. Production of some goods is only possible if a dedicated team of workers is assembled and trained in advance to make that particular good. Capacity to produce good j is determined by the number of people specializing in that good and an exogenously specified productivity. Given the precommitted specialized resources, the good can be produced at zero marginal cost up to the level of capacity.

Developing a new good is a costly gamble. But if it is successful, the team has a monopoly in producing the good and faces the linear demand curve shown in the top panel of Figure 1. The intercepts A_{jt} and $\bar{Q}_{jt}$ are functions of consumer preferences and the distribution of income that will be derived below. With zero marginal cost, the unit maximizes profits by producing up to the point where marginal revenue equals zero, namely the point $\bar{Q}_{jt}/2$, provided it has the capacity to do so.[1]

Suppose there is a shock to demand for good j that reduces both A_{jt} and $\bar{Q}_{jt}$ by a factor $\chi < 1$. In the model developed below, this would result from a change in consumer preferences that reduces the marginal utility of good j by the factor χ. The profit-maximizing response is for producers to lower the quantity of good j that is produced by the factor χ.

The change in preferences also implies an increase in the relative demand for other goods. But if those goods are already being produced at capacity, the profit-maximizing response is to raise the price of those items with no increase in quantity. If specialized resources cannot be costlessly shifted across activities, the result of a shock to relative demand is idle productive resources and a fall in total real GDP.

The decision to specialize is endogenous. Developing a new specialty takes time. Non-specialized workers evaluate the lifetime costs and benefits in making this decision. If the drop in demand is purely transitory, it does not effect any future returns, and thus does not affect the expected lifetime benefits of future specialization. There are some possible general equilibrium effects on impact, but in a calibration of the model these are quantitatively minor. The response to a purely transitory fall in demand for some goods is thus a temporary drop in total real output with essentially no lasting consequences.

Suppose instead that there is a change in preferences that increases A_{jt} and $\bar{Q}_{jt}$ by a factor $\chi > 1$. The profit-maximizing response of a unit that is already at capacity is to increase the price with no change in quantity. Apart from quantitatively modest general equilibrium effects, the increase in demand has no effect on real GDP. Empirical support for the proposition that a decrease in demand can have a bigger effect on output than an increase of the same magnitude was provided by Weise (1999) and Lo and Piger (2005). Scholars like Tobin (1972) and Ball and Mankiw (1994) attributed this observed asymmetry to the mechanics of partial price adjustment. Here, as in Murphy (2017) and Auerbach, Gorodnichenko, and Murphy (2020), the asymmetry is caused by technological costs of adjusting resources and would arise even with perfectly flexible prices. When demand falls below capacity, the profit-maximizing response is to lower both output and price, whereas an increase in demand above capacity leads only to a price increase.

If the shock to demand is persistent there can be interesting dynamic effects. In the technology modeled here, all members of a specialized team are indispensable. All members of

[1]The equation for a line with intercepts A_{jt} and $\bar{Q}_{jt}$ is $P_{jt} = A_{jt} - Q_{jt}A_{jt}/\bar{Q}_{jt}$ implying the revenue function $A_{jt}Q_{jt} - Q_{jt}^2 A_{jt}/\bar{Q}_{jt}$ with marginal revenue $A_{jt} - 2Q_{jt}A_{jt}/\bar{Q}_{jt}$. This is a line with the same vertical intercept as the demand curve and twice the slope. The horizontal intercept at $Q_{jt} = \bar{Q}_{jt}/2$ is the point at which elasticity of demand is unity.

the unit either remain committed to producing good j or else the unit disbands. If the drop in demand for good j is sufficiently large and long lasting, it may be in the lifetime interests of the workers to abandon production of the good and turn to producing a nonspecialized good or try to develop a new specialty. The number of nonspecialized workers at any time who are trying to develop a specialty is an endogenous response to the costs and benefits of specialization. Nonspecialized workers experience different i.i.d. shocks to their productivity at producing the nonspecialized good. Those with favorable productivity produce the nonspecialized good, while those with less favorable productivity try to develop a new specialty. The cut-off responds endogenously to economic conditions. But it is never the case that all workers are trying to specialize. The result is that it takes time for the economy to return to the level of specialization that is optimal in the long run. The gradualness of the return to the long-run equilibrium is a result of technological costs of reallocating resources, not any sluggishness in the adjustment of wages or prices.

In response to a drop in demand that is persistent but less severe, it may be in the interest of specialized workers to wait for demand for their specialty to recover. In this case there could be a longer-lasting episode of underutilized resources. The opportunities and rewards to specialization will be lower during the demand slump, resulting in a build-up over time in the fraction of the labor force without a specialization. This causes the consequences for total output to build over time, and can result in a hump-shaped response to the shock in which the maximum effect is not observed until many months after the initial shock. Empirical support and alternative explanations for a hump-shape response to demand were provided by Christiano, Eichenbaum and Evans (2005), Hamilton (2008), and Auclert, Rognlie and Straub (2020). Here the hump shape is an endogenous implication of the technological costs associated with reallocating specialized resources.

Figure 4 below illustrates a variety of dynamic responses that could result from a demand shock to 0.6% of GDP depending on the duration of the shock and how its incidence is distributed across different goods.

The core contribution of the paper is to explain how demand could be a key driver of short-run fluctuations but not matter for long-run fluctuations without relying on the assumption that wages or prices take time to adjust. The friction in this model that replaces the nominal rigidities in Keynesian models is the technological requirement that production of some goods requires specialized resources committed in advance.

There are of course many other papers that have proposed alternative explanations for how demand shocks could lead to lower output even with perfectly flexible prices. One popular approach interprets aggregate demand shocks as a decrease in desired current consumption relative to future consumption. However, Angeletos (2018) noted that this mechanism would predict recessions to be associated with higher investment and hours worked. An alternative literature emphasizes coordination problems. Models such as those developed by Cooper and

John (1988), Woodford (1991), Kaplan and Menzio (2016), and Fajgelbaum, Schaal, and Taschereau-Dumouchel (2017) are characterized by multiple equilibria, in which demand may or may not be a factor in aggregate fluctuations depending on equilibrium selection. Angeletos and Lian (2020) and Ilut and Saijo (2021) emphasized coordination of expectations in models with a unique equilibrium but in which individual actors are not fully rational. By contrast, the model developed here is characterized by a unique equilibrium in which everyone behaves completely rationally.

There is also a large literature that emphasizes sectoral shocks, costly reallocation, and mismatch. Shocks to productivity are the underlying cause of economic fluctuations in most of these models, such as Alvarez and Shimer (2011) and Şahin et al. (2014). Relative demand shocks are the driving variable in Guerrieri et al. (2021), but nominal wage and price rigidities are fundamental for propagating demand shocks in their model. Hamilton (1988) showed how sectoral demand shocks can lead to unemployment either from reallocation of labor or from impacted workers waiting for conditions in their sector to improve, but that was in a two-sector model without growth, creation of new goods, or monopoly power. None of these papers developed the case that costly reallocation alone could explain why demand shocks can have dynamic multiplier consequences for aggregate economic activity such as are typically attributed to nominal rigidities.

This paper makes a number of other contributions to the literature. It shows how monopoly power can be sustained in a growing economy even as new goods are introduced and some old goods are discontinued every period. It develops a new characterization of inequality as arising from successful gambles to create new goods. The costs associated with trying to create new goods determine steady-state income differentials and unemployment as well as the speed with which the economy recovers from shocks. The model allows for considerable heterogeneity, yet both individual and aggregate outcomes can be calculated using only a handful of equations. The model is also consistent with the observation that the unemployment rate has been remarkably stable despite a century of economic growth and technological innovations. Martellini and Menzio (2020) noted the challenges in explaining this using standard search and matching models and proposed an alternative explanation. In this model, a stable unemployment rate in the face of long-term economic growth is an equilibrium implication of the fact that the opportunity cost and potential benefits of being unemployed along with the tax base that finances compensation paid to the unemployed all grow with the overall level of productivity.

2 Demand for goods.

At time t the population consists of a continuum of individuals of measure N_t who each consume a discrete set $j \in \mathcal{J}_t$ of different goods. Goods are nonstorable, and there are no capital or financial markets, so that the budget constraint for individual i is

$$\sum_{j \in \mathcal{J}_t} P_{jt} q_{ijt} \leq y_{it} \tag{1}$$

where P_{jt} is the nominal price of good j, q_{ijt} is the quantity of good j consumed by individual i, and y_{it} the individual's nominal income. The objective of consumer i is to maximize[2]

$$U_{it} = \sum_{j \in \mathcal{J}_t} \frac{-\gamma_{ijt}}{2} (\bar{q}_{ijt} - q_{ijt})^2 \tag{2}$$

subject to (1).The first-order conditions for an interior solution are

$$\gamma_{ijt}(\bar{q}_{ijt} - q_{ijt}) = \lambda_{it} P_{jt} \qquad j \in \mathcal{J}_t \tag{3}$$

for λ_{it} the marginal utility of income. Holding λ_{it} constant, these imply a price elasticity of demand given by

$$\varepsilon_{ijt} = \left| \frac{\partial q_{ijt}}{\partial P_{jt}} \frac{\gamma_{ijt}(\bar{q}_{ijt} - q_{ijt})/\lambda_{it}}{q_{ijt}} \right| = \frac{\bar{q}_{ijt} - q_{ijt}}{q_{ijt}}. \tag{4}$$

Quadratic preferences have some important advantages for purposes of this model. As emphasized by Murphy (2017), quadratic preferences imply that the elasticity of demand changes as we move along the demand curve, which is important for understanding how decisions of monopolist producers respond to changing conditions. The price elasticity of consumer i's demand is less than one when $q_{ijt} > \bar{q}_{ijt}/2$ and greater than one when $q_{ijt} < \bar{q}_{ijt}/2$. In the general equilibrium described below, producers choose a level of production and price such that the market-wide elasticity is always greater than or equal to one. Along the steady-state growth path, the market-wide elasticity will turn out to be exactly equal to one.

Another advantage of quadratic utility over isoelastic preferences is that quadratic preferences allow the possibility that producers of j could be driven out of business if productivity or demand is too low. If the price P_{jt} becomes too high, a consumer with preferences (2) will choose $q_{ijt} = 0$, whereas isoleastic preferences imply that consumers always buy every good in equilibrium, willing to pay $P_{jt} \to \infty$ as $q_{ijt} \longrightarrow 0$. In the economy described below, some goods are always being discontinued and new goods are being created along the steady-state growth path.

A final advantage of quadratic preferences is that they result in simple closed-form solutions for key magnitudes.

Expenditure shares. A useful way to summarize the demand of individual i is by the fraction of income that consumer i chooses to spend on good j at time t. The following proposition

[2] If one were to motivate (2) as a second-order approximation to logarithmic preferences, $\bar{q}_{ijt}/2$ would be the value of q_{ijt} around which the Taylor expansion is taken. See the discussion of Figure 3 in Section 7.

gives some results that will prove useful in characterizing the equilibrium fraction of spending devoted to different goods along the steady-state growth path.

Proposition 1. *Define*

$$\alpha_{ijt} = \gamma_{ijt}(\bar{q}_{ijt}/2)^2. \tag{5}$$

(a) If prices and income are such that individual i would choose $q_{ijt} = \bar{q}_{ijt}/2$ for all $j \in J_t$, then the fraction of individual i's income that is spent on good j is proportional to α_{ijt}:

$$P_{jt}q_{ijt}/y_{it} = \alpha_{ijt}/\textstyle\sum_{j\in J_t}\alpha_{ijt}. \tag{6}$$

(b) If there is a set $\mathcal{M}_t^{(k)}$ of measure $R_t^{(k)}$ of different individuals at date t who all share the same preference parameters, that is, if $\gamma_{ijt} = \gamma_{jt}^{(k)}$ and $\bar{q}_{ijt} = \bar{q}_{jt}^{(k)}$for all $i \in \mathcal{M}_t^{(k)}$,and if prices and incomes are such that members of the group on average choose to consume $\bar{q}_{jt}^{(k)}/2$, that is, if

$$(1/R_t^{(k)})\int_{i\in\mathcal{M}_t^{(k)}} q_{ijt}di = \bar{q}_{jt}^{(k)}/2 \; \forall j \in J_t, \tag{7}$$

then the fraction of the group's income that is spent on good j is proportional to $\alpha_{jt}^{(k)}$:

$$\frac{\int_{i\in\mathcal{M}_t^{(k)}} P_{jt}q_{ijt}di}{\int_{i\in\mathcal{M}_t^{(k)}} y_{it}di} = \frac{\alpha_{jt}^{(k)}}{\sum_{j\in J_t}\alpha_{jt}^{(k)}}. \tag{8}$$

Market-wide demand curves. Summing across all consumers i gives the market demand curve $P_{jt} = A_{jt} - B_{jt}Q_{jt}$. Note we will be following the notational convention of using lower-case letters like q_{ijt} to refer to magnitudes for individual consumers i and upper case like Q_{jt} to refer to total magnitudes for individual goods j. Here $A_{jt} = \bar{Q}_{jt}/\Lambda_{jt}$, $B_{jt} = 1/\Lambda_{jt}$, $\Lambda_{jt} = \int_0^{N_t}(\lambda_{it}/\gamma_{ijt})di$ and

$$\bar{Q}_{jt} = \int_0^{N_t} \bar{q}_{ijt}di. \tag{9}$$

The marginal revenue for producers of good j is $MR_{jt} = A_{jt} - 2B_{jt}Q_{jt}$. The good-level elasticity has the same properties as the demand curves for individual consumers; the market-wide elasticity is greater than or equal to one provided $Q_{jt} \leq \bar{Q}_{jt}/2$.

3 Production of specialized goods.

Good $j = 1$ can be produced by anyone without any training or coordination with others. By contrast, goods $j > 1$ are specialized in the sense that their production requires a dedicated team who work together to produce the good. If any worker were to leave the team, the good could not be produced. Once the workers who form a team are assembled, they enjoy a monopoly in producing good j and base their production and pricing decisions on that monopoly power. Team j consists of a measure of N_{jt} workers and has total production capacity $X_{jt}N_{jt}$ where productivity per worker X_{jt} for the team evolves according to an exogenous process. At the time that its production and pricing decisions for period t are made, unit j takes X_{jt} and N_{jt} as given and chooses P_{jt} and Q_{jt} to maximize total profits $P_{jt}Q_{jt}$ subject to $P_{jt} = A_{jt} - B_{jt}Q_{jt}$, $P_{jt} \in [0, A_{jt}]$, and $Q_{jt} \leq X_{jt}N_{jt}$. The number of specialized goods is sufficiently large that unit j ignores the effect of its decisions on Λ_{jt} or the price and output of other units. The profit-maximizing strategy is to produce up to the point where marginal revenue equals zero if there is sufficient production capacity and to produce at production capacity if not:

$$Q_{jt} = \begin{cases} \bar{Q}_{jt}/2 & \text{if } X_{jt}N_{jt} \geq \bar{Q}_{jt}/2 \quad \text{[demand constrained]} \\ X_{jt}N_{jt} & \text{if } X_{jt}N_{jt} < \bar{Q}_{jt}/2 \quad \text{[supply constrained]} \end{cases} . \tag{10}$$

We will describe production of good j as demand constrained in the first instance and supply constrained in the second; see the top panel of Figure 1.

Note that under no circumstances would a monopolist ever choose to produce in the inelastic region of the demand curve. It is always the case for every period t and every specialized good j that $Q_{jt} \leq \bar{Q}_{jt}/2$.

New hiring. In period t, unit j takes its total capacity $N_{jt}X_{jt}$ as given. We assume that the hiring decision for $N_{j,t+1}$ is based on the goal of maximizing expected profit of the unit. Let $N^*_{j,t+1}$ denote the level of employment that maximizes expected revenue:

$$N^*_{j,t+1}E_t(X_{j,t+1}) = E_t(\bar{Q}_{j,t+1}/2). \tag{11}$$

Since the team could not be productive if any current member leaves, workers are not laid off even if $N^*_{j,t+1} < N_{jt}$. New workers are hired up to the level $N^*_{j,t+1}$ if $N^*_{j,t+1} > N_{jt}$. The number of positions offered to new members of the team who would begin working in $t + 1$ is thus $O_{jt} = \max\{N^*_{j,t+1} - N_{jt}, 0\}$.

Note that maximizing the profit of the ongoing unit is not the same objective as maximizing the income of continuing workers. We think of an observed firm as a collection of a large number of separate producing units, with the objective of the firm being to maximize total profit subject to the constraint that individuals are available to do the work at the offered terms. If instead we took the objective to be to maximize expected income of existing team members, that would add an additional friction to hiring in the model.

4 Nonspecialized workers.

We will refer to an individual who is not part of a specialized team at time t as "nonspecialized." Nonspecialized workers can choose between 3 options.

Option 1: seek to join an existing specialized unit. To pursue this option, an individual trains and applies in period t for a position to produce good j beginning in period $t+1$. With probability π_{jt} the individual will be successful. Each individual takes π_{jt} as given, though in equilibrium π_{jt} will be determined by the number of people applying for the job and the number of openings available. An individual who pursues this option will receive nominal compensation C_t while unemployed, financed through a proportional tax levied on the income of specialized workers during period t.

Option 2: seek to create a new good. An individual who is trying to join a team that creates a new good also receives unemployment compensation C_t during period t and has a probability k_π of being successful. There is also a utility cost k_U of making an effort to create a new good. The parameters k_π and k_U are fixed technological parameters that summarize the importance of frictions in creating new goods. If $k_\pi \to 1$ and $k_U \to 0$, the monopoly power of specialized teams would not be sustained along the steady-state growth path.

Option 3: produce good 1. Good 1 is assumed to be produced in a nonspecialized sector in which anyone could work with no training or coordination with others. If individual i works in sector 1, s/he could produce x_{it} units of good 1. The productivity parameter x_{it} is distributed independently across workers and across time. A favorable productivity x_{it} for individual i at time t has no implications for that same individual's productivity at $t + 1$. The nominal income of individual i during period t is given by

$$y_{it} = \begin{cases} P_{1t}x_{it} & \text{if produces good 1} \\ C_t & \text{if looks for a job} \end{cases} .$$

Objective of nonspecialized workers. Nonspecialized workers choose between the above three options, seeking to maximize

$$v_{it} = E_t \sum_{s=1}^{\infty} \beta^s \log y_{i,t+s} \tag{12}$$

where E_t denotes an expectation conditional on information available at date t and $0 < \beta < 1$ is a discount rate. We will motivate this objective as an approximation to expected lifetime utility in Section 7.

Let Y_{jt} be the after-tax nominal income of each individual who is part of specialized team j at date t,

$$Y_{jt} = (1 - \tau)P_{jt}Q_{jt}/N_{jt},$$

for τ the tax rate. Let V_{jt} denote the value of (12) for such an individual:

$$V_{jt} = \log Y_{jt} + \beta(1 - k_{jt})E_tV_{j,t+1} + \beta k_{jt}E_tV_{1,t+1}. \tag{13}$$

Here k_{jt} is the probability that unit j will discontinue production in $t + 1$. If the good is discontinued, next period those individuals will be nonspecialized. Since productivity x_{it} is drawn independently over time, the expected lifetime utility in the event that the team is disbanded is $E_t V_{1,t+1}$, the same for all individuals.

If a nonspecialized individual successfully creates a new good, the expected lifetime utility is $E_t V^\sharp_{t+1}$, whose value will be described below. Thus the value of (12) for a nonspecialized individual at time t is

$$v_{it} = \begin{cases} \log(P_{1t}x_{it}) + \beta E_t V_{1,t+1} & \text{if produces good 1} \\ \log C_t + \beta\pi_{jt} E_t V_{j,t+1} + \beta(1-\pi_{jt})E_t V_{1,t+1} & \text{if applies to join existing unit } j \\ \log C_t - k_U + \beta k_\pi E_t V^\sharp_{t+1} + \beta(1-k_\pi)E_t V_{1,t+1} & \text{if tries to create a new good} \end{cases} . \tag{14}$$

Decisions of nonspecialized workers. Individual i chooses the most favorable of the options in (14). The optimal decision is characterized by a productivity threshold X^*_{1t} such that individual i chooses to produce good 1 if $x_{it} \geq X^*_{1t}$ and looks for something better otherwise. If some individuals choose to produce good 1 and others try to create new goods, then X^*_{1t} would be the level of productivity at which the marginal nonspecialized individual is indifferent between working or trying to create a new good:

$$\log(P_{1t}X^*_{1t}) + \beta E_t V_{1,t+1} = \log C_t - k_U + \beta k_\pi E_t V^\sharp_{t+1} + \beta(1-k_\pi)E_t V_{1,t+1}. \tag{15}$$

Expression (15) can equivalently be written

$$\log(P_{1t}X^*_{1t}) - \log C_t = -k_U + \beta k_\pi E_t \tilde{V}^\sharp_{t+1} \tag{16}$$

where $\tilde{V}^\sharp_t = V^\sharp_t - V_{1t}$ is the expected lifetime advantage of specializing in a newly created good relative to being nonspecialized. Alternatively, when there is an incentive to try to specialize in continuing good j, (14) would require

$$\log(P_{1t}X^*_{1t}) + \beta E_t V_{1,t+1} = \log C_t + \beta\pi_{jt} E_t V_{j,t+1} + \beta(1-\pi_{jt})E_t V_{1,t+1} \tag{17}$$

$$\log(P_{1t}X^*_{1t}) - \log C_t = \beta\pi_{jt} E_t \tilde{V}_{j,t+1} \tag{18}$$

for $\tilde{V}_{jt} = V_{jt} - V_{1t}$ the lifetime advantage of specializing in j. In a typical equilibrium in which some individuals try to create a new good while others seek to join existing unit j, both conditions (16) and (18) hold, requiring that in equilibrium π_{jt} must satisfy

$$\beta\pi_{jt} E_t \tilde{V}_{j,t+1} = -k_U + \beta k_\pi E_t \tilde{V}^\sharp_{t+1}. \tag{19}$$

It follows from equations (14), (15), and (17) that the lifetime income of nonspecialized individual i is characterized by

$$v_{it} = \begin{cases} \log(P_{1t}x_{it}) + \beta E_t V_{1,t+1} & \text{if } x_{it} \geq X_{1t}^* \\ \log(P_{1t}X_{1t}^*) + \beta E_t V_{1,t+1} & \text{if } x_{it} < X_{1t}^* \end{cases}. \tag{20}$$

The expression $E_t V_{1,t+1}$ is the expected value for $v_{i,t+1}$ across individuals i. Since x_{it} is distributed independently across time, we can find the date t value of V_{1t} by taking the expected value of (20) across all nonspecialized individuals i at time t:

$$V_{1t} = \log(P_{1t}\tilde{X}_{1t}) + \beta E_t V_{1,t+1} \tag{21}$$

$$\log \tilde{X}_{1t} = P(x_{it} \geq X_{1t}^*) E[\log(x_{it}) | x_{it} \geq X_{1t}^*] + P(x_{it} < X_{1t}^*) \log X_{1t}^*. \tag{22}$$

Another object of interest is $\hat{X}_{1t}$, the average output of nonspecialized individuals:

$$\hat{X}_{1t} = E\left(x_{it} | x_{it} \geq X_{1t}^*\right) P(x_{it} \geq X_{1t}^*). \tag{23}$$

Note that this definition of $\hat{X}_{1t}$ means that if N_{1t} denotes the total number of nonspecialized individuals (including both those working and those unemployed), the total amount of good 1 that is produced is given by

$$Q_{1t} = N_{1t}\hat{X}_{1t}. \tag{24}$$

Distribution of productivity across nonspecialized workers. Simple closed-form expressions for key magnitudes can be obtained when log of productivity is distributed uniformly across nonspecialized workers.

Proposition 2. *Suppose that the log of potential productivity for producing good 1 is distributed independently across individuals as* $\log x_{it} \sim U(R_t, S_t)$ *and let* $\log X_{1t}^* \in [R_t, S_t]$ *be the threshold level of productivity above which nonspecialized individuals choose to produce good 1 (that is,* X_{1t}^* *satisfies (16) or (18)). Then:*

(a) the fraction of nonspecialized individuals who are employed is

$$h_{1t} = P(x_{it} \geq X_{1t}^*) = \frac{S_t - \log X_{1t}^*}{S_t - R_t}; \tag{25}$$

(b) the expected flow-equivalent productivity of nonspecialized individuals (value of (22)) is

$$\log \tilde{X}_{1t} = \frac{S_t^2 - 2R_t \log X_{1t}^* + (\log X_{1t}^*)^2}{2(S_t - R_t)} \tag{26}$$

which is monotonically increasing in X_{1t}^**;*

(c) the average output of nonspecialized individuals (expression (23)) is

$$\hat{X}_{1t} = \frac{\exp(S_t) - X_{1t}^*}{S_t - R_t}. \tag{27}$$

5 Entry and exit of specialized goods.

Preferences for newly created goods. The income advantage of a worker specialized in good j at time t is given by the fraction of income that consumers spend on good j divided by the fraction of the population specializing in producing j. The fraction of income that consumer i wants to spend on good j in turn depends on the preference parameters γ_{ijt} and $\bar{q}_{ijt}$.

We think of the creation of a new good as the discovery of something that is a new potential source of utility to consumers. This utility is governed by previously undiscovered preference parameters γ_{ijt} and $\bar{q}_{ijt}$. We assume that there exists a technology for making such discoveries with the feature that when a larger group of individuals join together to successfully create a particular new good, they are able to discover a good for which the preference parameters γ_{ijt} and $\bar{q}_{ijt}$ are such that consumers want to spend a larger share of their income on that good. A bigger team can discover a product that captures a larger market share. We characterize the preferences discovered for a new good j in terms of the share of income that consumers would want to spend on that good along the steady-state growth path. We greatly simplify the analysis by assuming that this share is the same across all consumers.

The steady-state growth path is characterized by an advantage to specialization that just compensates for the costs of trying to develop a specialty. We assume that newly created goods enter with this steady-state advantage.[3] This steady-state advantage, which we denote by ω^0, is a function of other technological parameters that will be derived below. Let q_{ijt}^0 denote the consumption of good j by individual i along the steady-state growth path. The steady-state growth path turns out to be characterized by $q_{ijt}^0 = \bar{q}_{ijt}/2$. From (5) this implies an expenditure share given by $\alpha_{ijt} = \gamma_{ijt}(q_{ijt}^0)^2$, which we assume is the same value α_j^0 across consumers and across time. If $J_{2t}^\sharp$ denotes the set of goods that are newly created in period t, the assumption that goods that are newly created at date t enter with the steady-state advantage ω^0 is thus represented by

$$\alpha_j^0 = \omega^0 n_{jt} \text{ for } j \in J_{2t}^\sharp. \tag{28}$$

Discontinued goods. A good will be discontinued if the expected benefit to workers from retaining that specialization is less than they could anticipate by returning to the pool of

[3]If new goods did not enter with the steady-state advantage, there would be additional dynamics introduced as the advantage for each individual new product converges to the steady-state value. This would complicate the characterization of dynamics without adding any additional insights.

nonspecialized workers:

$$\text{if } E_t V_{j,t+1} < E_t V_{1,t+1}, \text{ then } j \in \mathcal{J}_{2t}^{\flat}. \tag{29}$$

In Section 9 we will show how condition (29) could arise from a sufficiently severe shock to the demand for good j. We assume that shocks like this occur every period, causing a fraction k_X of goods to be discontinued every period along the steady-state growth path. Workers take this risk into account in assessing the potential benefits to specialization in (13) with $k_{jt} = k_X$ for every good j along the steady-state growth path.

Steady-state growth path. Along the steady-state growth path, the number of discontinued goods ($J_{2t}^{\flat}$) equals the number of newly created goods each period ($\mathcal{J}_{2t}^{\sharp}$) and the number of workers who successfully create new goods just balances population growth and the number induced to give up their previous specialization. The steady-state growth path is characterized by a constant fraction over time of the population without a specialty: $n_{1t} = n_1^0$.

Note that expression (28) does not restrict the number of goods or the relative expenditure shares of different specialized goods. Suppose for example that there are k_J new goods created each period and that the fraction of new-goods workers who produce good j is represented by a_{ℓ_j} with $a_1 + \cdots + a_{k_J} = 1$,

$$n_{jt} = a_{\ell_j} n_t^{\sharp} \text{ for } j \in \mathcal{J}_{2t}^{\sharp} \text{ and } \ell_j \in \{1, ..., k_J\} \tag{30}$$

where $n_t^{\sharp}$ is the fraction of the population that produce goods that were first created in period t. Then from (28), $\alpha_j^0 = \omega^0 a_{\ell_j} n_t^{\sharp}$ for $j \in \mathcal{J}_{2t}^{\sharp}$. With k_J goods newly created at t and $k_X J_{2t}$ goods discontinued (where J_{2t} is the number of specialized goods of all type produced at t), the number of goods produced along the steady-state growth path is given by the constant $J_2 = k_J / k_X$.

6 Steady-state growth with constant productivity.

In this section we consider an economy in which population grows at rate n and productivity is constant. Thus in this section the bounds on the productivity of nonspecialized workers R_t and S_t are constants R and S over time. The log of productivity of workers producing goods that are newly created at time t is drawn from a time-invariant distribution $\log X_{jt} \sim N(\mu, \sigma^2)$ for $j \in \mathcal{J}_{2t}^{\sharp}$ and the productivity of workers producing good j remains fixed as long as good j remains in production.

We assume that a constant fraction k_X of existing specialized goods is discontinued each period. Setting $k_{jt} = k_X$ in (13) and subtracting (21) from the result gives

$$\tilde{V}_{jt} = \log \tilde{Y}_{jt} + \beta(1 - k_X) E_t \tilde{V}_{t+1} \tag{31}$$

for $\tilde{V}_{jt} = V_{jt} - V_{1t}$ and $\tilde{Y}_{jt} = Y_{jt}/(P_{1t}\tilde{X}_{1t})$. We conjecture a steady-state growth path along which the share of spending on good 1 is constant at α_1 and $\tilde{Y}_{jt}$ is the same for all specialized goods and constant over time: $\tilde{Y}_{jt} = \tilde{Y}^0$. This would mean from (31) that the lifetime advantage of any specialization is constant: $\tilde{V}_{jt} = \tilde{V}^0$. We suppose that new goods enter with this same advantage. The steady-state growth path is also characterized by a constant fraction of the population that is nonspecialized ($n_{1t} = n_1^0$) and a constant fraction of the unemployed who are trying to acquire a specialty ($h_{1t} = h_1^0$). From (25)-(27) the latter would mean that X_{1t}^*, $\tilde{X}_{1t}$, and $\hat{X}_{1t}$ are constants. Also along the steady-state growth path, each good always has exactly the capacity to produce the profit-maximizing ouput,

$$Q_{jt} = N_{jt}X_{jt} = \bar{Q}_{jt}/2 \qquad j \in \mathcal{J}_{2t}, \tag{32}$$

with team j adding new workers as the population grows to achieve this. A fraction h_{0t} of the unemployed try to create new goods and the remaining $1 - h_{0t}$ apply for new openings with continuing goods, with $h_{0t} = h_0^0$ and the probability $\pi_t = \pi^0$ of a successful application both constant over time. In this section we prove the existence and uniqueness of such a steady-state growth path and sketch the forces that would cause an economy to converge to this path.

Advantage from specialization. A constant share of spending on good 1 would mean

$$\frac{\sum_{j\in\mathcal{J}_{2t}} P_{jt}Q_{jt}}{P_{1t}Q_{1t}} = \frac{1-\alpha_1}{\alpha_1} \tag{33}$$

for $\mathcal{J}_{2t}$ the set of specialized goods produced at t. Let Y_{st} denote the average after-tax income per person of specialized workers. From (33) and (24) this is

$$Y_{st} = \frac{(1-\tau)\sum_{j\in\mathcal{J}_{2t}} P_{jt}Q_{jt}}{(1-n_{1t})N_t} = \left[\frac{(1-\tau)(1-\alpha_1)}{\alpha_1}\right]\left[\frac{P_{1t}N_{1t}\hat{X}_{1t}}{(1-n_{1t})N_t}\right].$$

Let $\tilde{Y}_t$ be the ratio of Y_{st} to $P_{1t}\tilde{X}_{1t}$, the flow-equivalent income of nonspecialized in (22):

$$\tilde{Y}_t = \frac{Y_{st}}{P_{1t}\tilde{X}_{1t}} = \left[\frac{(1-\tau)(1-\alpha_1)n_{1t}}{\alpha_1(1-n_{1t})}\right]\frac{\hat{X}_1(X_{1t}^*)}{\tilde{X}_1(X_{1t}^*)} \tag{34}$$

where in light of (27) and (26) we have written $\hat{X}_{1t}$ and $\tilde{X}_{1t}$ as functions of X_{1t}^*. Note that if n_{1t} and X_{1t}^* are constant, then $\tilde{Y}_t$ is constant. Substituting (34) into (31), the steady-state advantage to specialization is

$$\tilde{V}(n_1^0, X_1^{*0}) = \left[\frac{1}{1-\beta(1-k_X)}\right]\left\{\log\left[\frac{(1-\tau)(1-\alpha_1)n_1^0}{\alpha_1(1-n_1^0)}\right] + \log\hat{X}_1(X_1^{*0}) - \log\tilde{X}_1(X_1^{*0})\right\}. \tag{35}$$

Creation of new goods. The numerator on the left side of (33) is the tax base, and from (24), the denominator is $P_{1t}N_{1t}\hat{X}_{1t}$. With a total of $(1-h_{1t})N_{1t}$ individuals collecting unemployment compensation, the compensation per individual is

$$C_t = \frac{\tau \sum_{j\in\mathcal{J}_{2t}} P_{jt}Q_{jt}}{(1-h_{1t})N_{1t}} = \left[\frac{\tau(1-\alpha_1)}{\alpha_1(1-h_{1t})}\right] P_{1t}\hat{X}_{1t}. \tag{36}$$

Let h_{Yt} denote the log difference between the income that the marginal nonspecialized individual could earn from producing good 1 and the income collected from unemployment compensation:

$$h_{Yt} = \log(P_{1t}X^*_{1t}) - \log C_t. \tag{37}$$

From expression (36) this is

$$h_{Yt} = -\log\left[\frac{\tau(1-\alpha_1)}{\alpha_1}\right] + \log(1-h_{1t}) + \log X^*_{1t} - \log \hat{X}_{1t} = h_Y(X^*_{1t}). \tag{38}$$

We can write the equilibrium condition for creation of new goods (16) as

$$h_{Yt} = -k_U + k_\pi \beta \tilde{V}_t.$$

The steady-state solution is

$$-\log\left[\frac{\tau(1-\alpha_1)}{\alpha_1}\right] + \log(1-h_1(X_1^{*0})) + \log X_1^{*0} - \log \hat{X}_1(X_1^{*0}) = -k_U + k_\pi\beta\tilde{V}(n_1^0, X_1^{*0}). \tag{39}$$

New hiring. If each continuing good adds workers at the rate of population growth, along the steady-state growth path the total number of new openings is

$$O_t = (1-k_X)(e^n - 1)(1-n_{1t})N_t. \tag{40}$$

Since each continuing good offers the same lifetime advantage, the probability of successfully applying for one of these positions is the same across all continuing goods. With $(1-h_{1t})(1-h_{0t})n_{1t}N_t$ individuals applying for these positions, the probability of success is

$$\pi_t = \frac{(1-k_X)(e^n-1)(1-n_{1t})}{(1-h_{1t})(1-h_{0t})n_{1t}} = \pi(n_{1t}, X^*_{1t}, h_{0t}). \tag{41}$$

Individuals are indifferent between applying for existing jobs and trying to create new goods when (19) holds:

$$-k_U + k_\pi\beta\tilde{V}(n_1^0, X_1^{*0}) = \pi(n_1^0, X_1^{*0}, h_0^0)\beta\tilde{V}(n_1^0, X_1^{*0}). \tag{42}$$

Changes in the number of specialized workers. Note that $(1-h_{1t})h_{0t}k_\pi n_{1t}N_t$ individuals will join newly created units in $t+1$ which would be added to the $(1-k_X)(1-n_{1t})e^nN_t$ workers at continuing units. The total number of nonspecialized at $t+1$, which could be written as $n_{1,t+1}e^n N_t$, would then consist of the total population at $t+1$ ($e^n N_t$) minus the total number of specialized individuals:

$$n_{1,t+1}e^n N_t = e^n N_t - (1-h_{1t})h_{0t}k_\pi n_{1t}N_t - (1-k_X)(1-n_{1t})e^n N_t$$

$$n_{1,t+1} = n_{1t} + k_X(1-n_{1t}) - e^{-n}h_{0t}(1-h_{1t})k_\pi n_{1t}. \tag{43}$$

Thus the fraction of nonspecialized workers will be constant if

$$k_X(1-n_1^0) = e^{-n}h_0^0 k_\pi(1-h_1(X_1^{*0}))n_1^0. \tag{44}$$

The conditions for steady-state growth are characterized by the three equations (44), (42) and (39) in the three unknowns X_1^{*0}, n_1^0, h_0^0.

Proposition 3. *If $k_\pi, k_X, \alpha_1, \beta, \tau$ are all $\in (0,1)$ and n and k_U are both positive, there exists a unique value of (X_1^{*0}, n_1^0, h_0^0) for which (44), (42) and (39) simultaneously hold. At this solution,* $\log X_1^{*0} \in (R,S)$, $h_Y(X_1^{*0}) > 0$, $\tilde{V}(n_1^0, X_1^{*0}) > 0$, and $0 < \pi(n_1^0, X_1^{*0}, h_0^0) < 1$.

The fact that $\tilde{V}^0$ is positive means that individuals would prefer to be specialized if they could acquire a specialty at no cost. The barriers to becoming specialized (a probability k_π less than one of being able to join an existing enterprise and a cost k_U of trying to create a new one) require as compensation that $\tilde{V}^0$ be positive in equilibrium. The value of ω^0 is given by $(1-\alpha_1)/(1-n_1^0)$. Letting n_j^0 denote the value of n_{jt} when good j was first introduced ($n_j^0 = n_{jt}$ for $j \in J_{2t}^\sharp$), expression (28) becomes

$$\alpha_j^0 = n_j^0 \frac{(1-\alpha_1)}{(1-n_1^0)}. \tag{45}$$

Converging to the steady state. Figure 2 plots $\pi_t\beta\tilde{V}_{t+1}$ and $-k_u + k_\pi\beta\tilde{V}_{t+1}$ as functions of $\tilde{V}_{t+1}$. The point at which nonspecialized individuals are indifferent between applying for an existing job and trying to create a new good is the point at which the two lines cross. If the advantage to specialization is a large value like $\tilde{V}_{t+1}^{[1]}$, then π_t must be a large value like $\pi_t^{[1]}$ shown in the figure. For a given level of job openings O_j, this means from (41) that $(1-h_{0t})$ is small and the fraction of unemployed seeking to create new products h_{0t} is bigger. Thus a large value of $\tilde{V}_{t+1}$ encourages more people to try to create new goods, which from (43) means that $n_{1,t+1}$ will be lower because more people will develop a specialty. From (35), a lower value of n_1 will bring $\tilde{V}$ down. The steady state is characterized by a value $\tilde{V}^0$ for which $n_{1,t+1} = n_{1t} = n_1^0$.

7 Steady-state growth with growing productivity.

Here we generalize to an economy in which productivity grows at the rate g. Proposition 4 establishes that if the threshold X_{1t}^* grows at the same rate g as overall productivity and if the fraction of the population without a specialization is constant, then the unemployment rate, the income advantage to specialization, and the flow value of searching for work are all constant.

Proposition 4. *Let $h_{1t}, \tilde{X}_{1t}, \hat{X}_{1t}, C_t, h_{Yt}$, and $\tilde{Y}_t$ be given by (25)-(27), (36), (37) and (34). If $R_{t+1} = R_t + g$, $S_{t+1} = S_t + g$, and $\log X_{1,t+1}^* = \log X_{1t}^* + g$, then $h_{1,t+1} = h_{1t}$, $\log \tilde{X}_{1,t+1} = \log \tilde{X}_{1t} + g$, $\log \hat{X}_{1,t+1} = \log \hat{X}_{1t} + g$, $\log C_{t+1} = \log C_t + g$, and $h_{Y,t+1} = h_{Yt}$. If also $n_{1,t+1} = n_{1t}$, then $\tilde{Y}_{t+1} = \tilde{Y}_t$.*

Quadratic preferences in a growing economy. We assume for a growing economy that creation of a new good by $n_j^0 N_t$ people at date t arises from the discovery of a good on which individuals will want to consume a constant fraction $\alpha_j^0 = \omega^0 n_j^0$ of their growing income along the steady-state growth path. A useful way to think about this is to relate our assumption of quadratic preferences to a specification in which individual i has a logarithmic utility function:

$$U_{it}^\dagger = \sum_{j \in \mathcal{J}_t} \alpha_{jt} \log q_{ijt}. \tag{46}$$

Let q_{ijt}^0 denote the consumption of good j by individual i at date t along the steady-state growth path. A second-order approximation to (46) around the steady-state growth path gives

$$\begin{aligned} U_{it}^\dagger &\simeq \sum_{j \in \mathcal{J}_t} \alpha_{jt} \left[\log q_{ijt}^0 + \frac{1}{q_{ijt}^0}(q_{ijt} - q_{ijt}^0) - \frac{1}{2\left(q_{ijt}^0\right)^2}(q_{ijt} - q_{ijt}^0)^2 \right] \\ &= \sum_{j \in \mathcal{J}_t} \left[\delta_{ijt}^0 - \frac{\gamma_{ijt}}{2}(\bar{q}_{ijt} - q_{ijt})^2 \right] \end{aligned}$$

$$\bar{q}_{ijt} = 2q_{ijt}^0 \tag{47}$$

$$\gamma_{ijt} = \frac{\alpha_{jt}}{\left(q_{ijt}^0\right)^2}. \tag{48}$$

See the top panel of Figure 3. Note that if consumption on the steady-state growth path were $q_{ijt}^0 = \bar{q}_{ijt}/2$, expression (48) would just be another way to write expression (5).

A typical approach in macroeconomics is to assume that an expression like (46) is the true utility function and (2) an approximation in which the approximating parameters $\bar{q}_{ijt}$ and γ_{ijt} are functions of the true preference parameter α_{jt} and the steady-state consumption q_{ijt}^0. Under that interpretation, the expenditure share would be exactly equal to α_{jt} and the elasticity would be be exactly equal to one for all t. By contrast, here we are proposing to view

(2) as the true preferences and (46) as an approximation. Under this interpretation, in the neighborhood of the steady state, the expenditure share is approximately α_{jt} and the elasticity is approximately one. This second interpretation allows us to incorporate the benefits of using quadratic preferences noted in Section 2 while also taking advantage of some of the attractive steady-state growth features of logarithmic preferences.

Note that maximization of (46) subject to the budget constraint (1) has the solution $P_{jt}q_{ijt} = \alpha_{jt}y_{it}$ Substituting this into (46) gives the indirect utility function,

$$\sum_{j\in\mathcal{J}_t} \alpha_{jt} \log\left(\frac{\alpha_{jt}y_{it}}{P_{jt}}\right) = \log y_{it} + \sum_{j\in\mathcal{J}_t} \alpha_{jt} \log \alpha_{jt} - \sum_{j\in\mathcal{J}_t} \alpha_{jt} \log P_{jt},$$

which provides a motivation for (12).

Assumptions behind the steady-state growth path with productivity growth. Population grows at a fixed rate n starting from a value N_{t_0} at an initial date t_0. Log productivity at date t_0 is distributed $\log x_{it_0} \sim U(R_{t_0}, S_{t_0})$ across individuals i for nonspecialized workers and $\log X_{jt_0} \sim N(\mu, \sigma^2)$ across goods j for specialized workers producing good j. There are k_J types of specialized goods at date t_0 as in (30). Each period a fraction k_X of each type of good is discontinued and one new good of each type is created. The log of the productivity of newly created goods at date t is drawn from the distribution $\log X_{jt} \sim N(\mu + gt, \sigma^2)$ for $j \in \mathcal{J}^{\sharp}_{2t}$ and $t = t_0 + 1, t_0 + 2, ...$ with $\log X_{j,t+1} = \log X_{j,t} + g$ for as long as specialized good j is produced. Likewise $\log x_{it} \sim U(R_t, S_t)$ with $R_{t+1} = R_t + g$ and $S_{t+1} = S_t + g$ for $t = t_0, t_0 + 1, ...$ Each consumer wants to spend a fraction α_j^0 of their income along the steady-state growth path on good j as long as the good remains produced and the desired share of spending on good 1 is constant at α_1 along the steady-state growth path.

Proposition 5. *Let (X_1^{*0}, n_1^0, h_0^0) be the unique solution to (44), (42) and (39) for $R = R_{t_0}$ and $S = S_{t_0}$. If at initial date t_0 there are $1/k_X$ specialized goods of each type a_k (so that the initial total number of goods is $J_{2,t_0} = k_J/k_X$), the initial share of nonspecialized workers is $n_{1t_0} = n_1^0$, and the initial share of the population specialized in good j satisfies*

$$n_{jt_0} = \frac{\alpha_j^0(1 - n_1^0)}{1 - \alpha_1} \qquad j \in \mathcal{J}_{2t_0}, \tag{49}$$

then for all $t \geq t_0$:

(a) the fraction of the population that is nonspecialized, the threshold at which nonspecialized choose unemployment, the fraction of the nonspecialized who are employed, and fraction of the nonspecialized who try to create new goods are constant over time,

$$n_{1t} = n_1^0 \quad X_{1t}^* = X_1^{*0} \quad h_{1t} = h_1^0 \quad h_{0t} = h_0^0;$$

(b) the number of specialized goods in production is constant: $J_{2t} = k_J/k_X$;

(c) the consumption of good j by every specialized worker at time t is given by

$$q^0_{sjt} = \frac{(1-\alpha_1)(1-\tau)}{1-n^0_1} n^0_j X_{jt} \quad j \in \mathcal{J}_t \tag{50}$$

where X_{1t} is defined to be $\hat{X}_{1t}$;

(d) the average consumption of good j by nonspecialized workers at time t is given by

$$q^0_{njt} = \frac{[\alpha_1 + \tau(1-\alpha_1)]}{n^0_1} n^0_j X_{jt} \quad j \in \mathcal{J}_t; \tag{51}$$

(e) the share of the population that produces good j remains constant as long as the good remains in production: $n_{jt} = n^0_j$ for $j \in \mathcal{J}_t$ and the quantity of any good grows at rate $n+g$ for as long as it is produced:

$$\log Q_{j,t+1} = n + g + \log Q_{jt} \quad j \in \{\{1\} \cup \mathcal{J}^{\natural}_{2,t+1}\};$$

(f) the relative price of good j at time t is given by

$$p_{jt} = \frac{P_{jt}}{P_{1t}} = \frac{\alpha_j n^0_1 \hat{X}_{1t}}{\alpha_1 n^0_j X_{jt}} \tag{52}$$

which is constant over time as long as the good continues to be produced;

(g) the total demand parameter for good j is given by

$$\frac{\bar{Q}^0_{jt}}{2} = [n^0_1 q^0_{njt} + (1-n^0_1) q^0_{sjt}] N_t \quad j \in \mathcal{J}_t;$$

(h) at any date t, all specialized workers earn the same income as each other and the log difference between their income and that of the average nonspecialized is a constant over time.

8 Adjustment dynamics.

In this section we analyze the path by which an economy that is not in steady state converges over time to the steady-state growth path. We begin by parameterizing departures from steady state.

Shocks to demand and supply. In Section 7 we assumed that if a fraction n^0_j of the population successfully creates a new good j, they discover a good for which the preference parameters for individual i at date t are given by $\bar{q}_{ijt} = 2q^0_{ijt}$ and $\gamma_{ijt} = \alpha^0_j/(q^0_{ijt})^2$ where q^0_{ijt} is the consumption of good j by individual i along the steady-state growth path and $\alpha^0_j = n^0_j(1-\alpha_1)/(1-n^0_1)$. We now generalize this to study an economy in which the preference parameters for individual i at date t are given by $\bar{q}_{ijt} = \chi_{jt} 2q^0_{ijt}$ and $\gamma_{ijt} = \xi_{jt}\alpha_{jt}/(q^0_{ijt})^2$. Here $\chi_{jt} > 1$ would represent the possibility that consumers value good j more than normal at date t while $\chi_{jt} < 1$ would represent lower than normal demand. Since the market-wide demand parameter $\bar{Q}_{jt}$ is given by $\int_0^{N_t} \bar{q}_{ijt}di$, an increase in χ_{jt} shifts the demand for good j up. The magnitude ξ_{jt} is a different demand shock that changes A_{jt} without changing $\bar{Q}_{jt}$ in Figure 1. Thus ξ_{jt} makes the demand curve steeper or flatter which will lead producers to change the price but does not change the profit-maximizing level of output. Without loss of generality[4] we assume that $\chi_{1t} = \xi_{1t} = 1$.

To understand the role of α_{jt}, recall from Proposition 1 that if all specialized goods were producing at the unconstrained profit-maximizing level ($Q_{jt} = \bar{Q}_{jt}/2$ for all $j \in J_{2t}$) and if $\sum_{j \in J_t} \alpha_{jt} = 1$, then $\sum_{j \in J_{2t}} \alpha_{jt}$ would measure the share of spending on specialized goods. Although each producer has a monopoly in their particular good, if $\sum_{j \in J_{2t}} \alpha_{jt}$ is constant at $1 - \alpha_1$ for all t, specialists are always competing with each other for a fixed share $(1 - \alpha_1)$ of consumers' budgets. We adopt a simple characterization in which the share of good j out of the total share of specialized goods is determined by the share of producers of that good relative to all specialists:

$$\alpha_{jt} = \frac{n_{jt}}{(1 - n_{1t})}(1 - \alpha_1) \quad \text{for } j \in J_{2t}. \tag{53}$$

Note (53) ensures that $\sum_{j \in J_{2t}} \alpha_{jt} = 1 - \alpha_1$ for all t and simplifies to the earlier specification $\alpha_{jt} = \alpha_j^0$ in (45) if the economy is on the steady-state growth path at date t. Expression (53) implies that if off the steady-state growth path more people than usual become new specialists, they take some of the short-run expenditure share away from existing specialized goods. Again note that the specification of α_{jt} has no effect on quantity Q_{jt} that producers of good j choose to produce, but does influence the price they charge for the good and thus the income that producers receive off the steady-state growth path.

Finally, we specify the productivity of producers of good j as $X_{jt} = \zeta_{jt} X_{jt}^0$ where X_{jt}^0 is the productivity associated with the steady-state growth path. Here $\zeta_{jt} > 1$ captures a favorable productivity shock at date t and $\zeta_{jt} < 1$ represents lower than normal productivity.

We now describe the equations of motion for an economy that begins at date t_0 in which the initial values of n_{jt_0} and n_{1t_0} may not be at the steady-state values n_j^0 and n_1^0 and the values of χ_{jt}, ξ_{jt}, and ζ_{jt} may not equal unity for a finite number of initial periods $t_0, t_0 + 1, ..., t_0 + D$.

Proposition 6. *At any point off the steady-state growth path:*

(a)

$$\bar{Q}_{jt}/2 = \chi_{jt} H_t n_j^0 X_{jt}^0 N_t = \chi_{jt} H_t Q_{jt}^0 \quad \textit{for } j \in \mathcal{J}_t \tag{54}$$

$$H_t = 1 + \lambda_H (n_{1t} - n_1^0) \tag{55}$$

$$\lambda_H = \frac{\alpha_1 + \tau(1 - \alpha_1) - n_1^0}{n_1^0(1 - n_1^0)} \tag{56}$$

[4] A lower demand for good 1 could equivalently be expressed as $\xi_{jt} > 1 \, \forall j \in \mathcal{J}_{2t}$.

with $\lambda_H < 0$ for typical parameter values;

(b) the quantity of good j that is produced at date t is

$$Q_{jt} = \begin{cases} n_{1t} N_t \hat{X}_{1t} & \text{for } j = 1 \\ \min\{\bar{Q}_{jt}/2, n_{jt} N_t \zeta_{jt} X^0_{jt}\} & \text{for } j \in \mathcal{J}_{2t} \end{cases}; \tag{57}$$

(c) the relative price of good j at date t is

$$p_{jt} = \frac{P_{jt}}{P_{1t}} = \left(\frac{P^0_j}{P^0_1}\right)^2 \left(\frac{\alpha_1}{\alpha^0_j}\right) \left(\frac{\alpha_{jt}}{\alpha^0_j}\right) \xi_{jt} \left[\frac{\bar{Q}_{jt} - Q_{jt}}{\bar{Q}_{1t} - Q_{1t}}\right] \quad \text{for } j \in \mathcal{J}_{2t} \tag{58}$$

which in the special case when $Q_{jt} = \bar{Q}_{jt}/2$ becomes

$$p_{jt} = \frac{P_{jt}}{P_{1t}} = \xi_{jt} \left(\frac{P^0_j}{P^0_1}\right) \left(\frac{n_{jt}(1 - n^0_1)}{n^0_j (1 - n_{1t})}\right) \left[\frac{\chi_{jt} H_t n^0_1 \hat{X}^0_{1t}}{2 H_t n^0_1 \hat{X}^0_{1t} - n_{1t} \hat{X}_{1t}}\right]; \tag{59}$$

(d) the share parameter for good j is characterized by

$$\alpha_{jt} = \frac{n_{jt}(1 - n^0_1)}{n^0_j (1 - n_{1t})} \alpha^0_j \quad \text{for } j \in \mathcal{J}_{2t}; \tag{60}$$

(e) if good j continues into $t+1$, the number of individuals specializing in j at $t+1$ is

$$N_{j,t+1} = \max\left\{\frac{\bar{Q}_{j,t+1}}{2 X_{j,t+1}}, N_{jt}\right\} \quad \text{for } j \in \mathcal{J}^\natural_{2,t+1}, \tag{61}$$

the overall fraction of continuing specialists is

$$n^\natural_{t+1} = \sum\nolimits_{j \in \mathcal{J}^\natural_{2,t+1}} N_{j,t+1}/N_{t+1},$$

and if $N_{j,t+1} = \bar{Q}_{j,t+1}/(2X_{j,t+1})$ then

$$\frac{N_{j,t+1}}{N^0_{j,t+1}} = \frac{\chi_{j,t+1} H_{t+1}}{\zeta_{j,t+1}}; \tag{62}$$

(f) after-tax income per individual specialized in good j ($Y_{jt} = (1-\tau) P_{jt} Q_{jt}/N_{jt}$) is characterized by

$$\frac{Y_{jt}}{P_{1t}} = \frac{Y^0_t \xi_{jt} (1 - n^0_1) Q_{jt} (\bar{Q}_{jt} - Q_{jt}) Q^0_{1t}}{(1 - n_{1t})(Q^0_{jt})^2 (\bar{Q}_{1t} - Q_{1t})} \quad \text{for } j \in \mathcal{J}_{2t} \tag{63}$$

which if $Q_{jt} = \bar{Q}_{jt}/2$ simplifies to

$$\frac{Y_{jt}}{P_{1t}} = \frac{Y^0_t \xi_{jt} (1 - n^0_1)(\chi_{jt} H_t)^2 Q^0_{1t}}{(1 - n_{1t})(2 H_t Q^0_{1t} - n_{1t} \hat{X}_{1t} N_t)} \tag{64}$$

and which along the steady-state growth path is the same for all specialized workers:

$$Y_t^0 = \frac{(1-\tau)(1-\alpha_1)n_1^0}{\alpha_1(1-n_1^0)}\hat{X}_{1t}^0; \tag{65}$$

(g) compensation per unemployed individual is

$$C_t = \frac{\tau \sum_{j\in\mathcal{J}_{2t}} n_{jt}Y_{jt}}{n_{1t}(1-h_{1t})(1-\tau)}; \tag{66}$$

(h) the lifetime advantage of being specialized in good j relative to being nonspecialized is

$$\tilde{V}_{jt} = \log Y_{jt} - \log(P_{1t}\tilde{X}_{1t}) + \beta(1-k_X)\tilde{V}_{j,t+1} \quad \textit{for } j \in \mathcal{J}_{2t} \tag{67}$$

with good j endogenously discontinued after period t ($j \in \mathcal{J}_t^{\flat}$) if $\tilde{V}_{j,t+1} < 0$;

(i) if some individuals spend period t trying to create a new good, then

$$\log(P_{1t}X_{1t}^*) - \log C_t = -k_U + k_\pi\beta\tilde{V}_{j,t+1} \quad \textit{for } j \in \mathcal{J}_{2,t+1}^{\sharp}; \tag{68}$$

(j) if a positive fraction h_{jt} of unemployed workers seek to specialize in continuing good j, the fraction π_{jt} who are successful is characterized by

$$\pi_{jt} = \frac{N_{j,t+1} - N_{jt}}{(1-h_{1t})h_{jt}n_{1t}N_t}$$

$$\log(P_{1t}X_{1t}^*) - \log C_t = \pi_{jt}\beta\tilde{V}_{j,t+1} \tag{69}$$

for $j \in \mathcal{J}_{2,t+1}^{\natural}$ and $h_{0t} = 1 - \sum_{j\in\mathcal{J}_{2,t+1}^{\natural}} h_{jt}$ the fraction seeking to create new goods;

(k) the fraction of the population in $t+1$ producing newly created goods is

$$n_{t+1}^{\sharp} = e^{-n}(1-h_{1t})h_{0t}n_{1t}k_\pi \tag{70}$$

and the fraction of the population that is nonspecialized is given by

$$n_{1,t+1} = 1 - n_{t+1}^{\sharp} - n_{t+1}^{\natural}. \tag{71}$$

(l) Define real GDP to be the ratio of current production evaluated at steady-state prices to steady-state production evaluated at steady-state prices:

$$Q_t = \frac{\sum_{j\in\mathcal{J}_t} P_{jt}^0 Q_{jt}}{\sum_{j\in\mathcal{J}_t} P_{jt}^0 Q_{jt}^0}. \tag{72}$$

This can equivalently be written as

$$Q_t = \sum\nolimits_{j \in \mathcal{J}_t} \alpha_j^0 (Q_{jt}/Q_{jt}^0) = \left(\frac{1-\alpha_1}{1-n_1^0}\right) \sum\nolimits_{j \in \mathcal{J}_{2t}} \left(\frac{Q_{jt}}{N_t X_{jt}^0}\right) + \left(\frac{\alpha_1}{n_1^0}\right) \left(\frac{\hat{X}_{1t}}{\hat{X}_{1t}^0}\right) n_{1t}. \tag{73}$$

In the special case when all goods are demand constrained and $\chi_{jt} = \zeta_{jt} = 1 \ \forall j$*, this becomes*

$$Q_t = \frac{H_t(1-\alpha_1)}{1-n_1^0} \sum\nolimits_{j \in \mathcal{J}_{2t}} n_j^0 + \left(\frac{\alpha_1}{n_1^0}\right) \left(\frac{\hat{X}_{1t}}{\hat{X}_{1t}^0}\right) n_{1t}. \tag{74}$$

The role of demand in determining real output. Recall from Proposition 5e that the long-run growth rate of real output is determined solely by growth in population and productivity. By contrast, Proposition 6 identifies demand as potentially important in the short run. Whereas Keynesian models attribute this difference between short and long run to the time necessary for wages and prices to adjust to economic conditions, here it arises solely from the time required for productive resources to be reallocated. For example, if the current number of specialists in good j is higher than warranted by long-run demand ($n_{jt_0} > n_j^0$), long-run equilibrium is eventually restored either by limiting new hires in j until population growth returns the share to n_j^0 or, if the rewards from waiting for a return to profitability are insufficient, discontinuing production of j altogether. In the latter case, some of the individuals will eventually develop a new specialty, but doing so takes time. If the excess supply results instead from a temporary drop in demand ($\chi_{jt} < 1$ for $t = t_0, t_0+1, ..., t_0+D$), the result would again be a temporary drop in output with specialized factors of production underutilized as they wait for demand conditions to improve, or again possibly a permanent discontinuation of the good's production if demand is expected to remain depressed for a sufficiently long period. Shortfalls in demand can have effects that are long lasting as a result of the technological costs of developing new specialties, but will not have permanent effects because eventually specialties will adapt to long-run incentives.

Result 6a establishes that the overall number of nonspecialized individuals n_{1t} is itself a factor entering demand for specialized goods. An increase in n_{1t} results in lower total demand provided that $\alpha_1 + \tau(1-\alpha_1) < n_1^0$. In interpreting this inequality, note that α_1 is the steady-state fraction of income that goes to nonspecialized individuals as a result of production of good 1 and $\tau(1-\alpha_1)$ is the fraction collected as unemployment compensation. If the sum of these is less than n_1^0, the fraction of the population that is nonspecialized, then the average after-tax income of a nonspecialized individual along the steady-state path is less than that of someone who is specialized. This is all that is needed to conclude that $\lambda_H < 0$. This condition is almost guaranteed by Proposition 3, which established that $\tilde{V}^0 > 0$, meaning that the log after-tax income of specialized workers exceeds the expected log income of nonspecialized along the steady-state growth path. However, because of Jensen's Inequality, this is not quite enough to conclude that expected specialized after-tax income also exceeds the expected income of the nonspecialized, which is the condition required by $\alpha_1 + \tau(1-\alpha_1) < n_1^0$. For most parameter values, Jensen's Inequality is not big enough to reverse the typical outcome. Online Appendix D provides sufficient conditions under which λ_H is necessarily negative. When $\lambda_H < 0$, $\bar{Q}_{jt}$ is lower when the fraction of nonspecialized individuals is higher.

Short-run determinants of real GDP. Note that we calculated real GDP in (72) as the ratio of current to steady-state output evaluated at steady-state prices. Thus $Q_t > 1$ means a value of real GDP higher than steady state and $Q_t < 1$ means a value lower than steady state. As an example, consider the special case when there are no demand or productivity shocks ($\chi_{jt} = \zeta_{jt} = 1$), all goods have capacity to produce the profit-maximizing output ($Q_{jt} = \bar{Q}_{jt}/2$), and the population share of each specialty is the steady-state value ($n_{jt} = n_j^0$). In this case (74) becomes

$$Q_t = H_t \left(\frac{1-\alpha_1}{1-n_1^0}\right)(1-n_{1t}) + \left(\frac{\alpha_1}{n_1^0}\right)\left(\frac{\hat{X}_{1t}}{\hat{X}_{1t}^0}\right) n_{1t} \tag{75}$$

Note that $(1-\alpha_1)/(1-n_1^0)$, is greater than 1 and (α_1/n_1^0) is less than 1. Thus when $H_t = (\hat{X}_{1t}/\hat{X}_{1t}^0) = 1$,

$$\frac{\partial Q_t}{\partial n_{1t}} = \frac{\alpha_1}{n_1^0} - \frac{1-\alpha_1}{1-n_1^0} < 0.$$

Thus even if H_t and $(\hat{X}_{1t}/\hat{X}_{1t_0})$ were unity, a higher fraction of nonspecialized workers would mean lower GDP because fewer of the goods that consumers value are being produced. When $n_{1t} > n_1^0$, both $(\hat{X}_{1t}/\hat{X}_{1t}^0) < 1$ because when more individuals are nonspecialized, a higher fraction of them look for jobs,[5] and also $H_t < 1$ due to lower demand. Both these are additional factors pushing real GDP below 1 when $n_{1t} > n_1^0$. A higher number of nonspecialized lowers real GDP.

Linearized adjustment dynamics. We can get some additional understanding by linearizing the results in Proposition 6. Let $w_t^\dagger$ denote the deviation of the variable w_t or its log from the value on the steady-state growth path.[6] Online Appendix B shows that if j is a continuing specialized good that is demand constrained in period t (that is, if $j \in J_{2t}^\dagger$, $t = t_0+1, t_0+2, ...$, and $\bar{Q}_{jt}/2 < n_{jt}N_tX_{jt}$), then the deviations from steady state of output, employment, and relative price are characterized by

$$Q_{jt}^\dagger = \chi_{jt}^\dagger + \lambda_H n_{1t}^\dagger \tag{76}$$

$$n_{jt}^\dagger = n_{j,t-1}^\dagger - n \tag{77}$$

[5] If $n_{1t} > n_1^0$, then $X_{1t}^* > X_{1t}^{*0}$ and $\hat{X}_{1t} < \hat{X}_{1t}^0$.

[6] Specifically, $w_t^\dagger = \log w_t - \log w_t^0$ for $w_t = Q_{jt}, X_{1t}^*, \chi_{jt}, \xi_{jt}, \zeta_{jt}$; $w_t^\dagger = w_t - w^0$ for $w_t = n_{jt}$; and $p_{jt}^\dagger = \log(P_{jt}/P_{1t}) - \log(P_{jt}^0/P_{1t}^0)$.

$$p_{jt}^{\dagger} = \xi_{jt}^{\dagger} + \chi_{jt}^{\dagger} + \lambda_H n_{1t}^{\dagger} + \frac{1}{n_1^0(1-n_1^0)} n_{1t}^{\dagger} + \frac{n_{jt}^{\dagger}}{n_j^0} + \lambda_5 X_{1t}^{*}. \tag{78}$$

An increase in demand that shifts the demand curve up ($\chi_{jt}^{\dagger} > 0$) leads to an increase in production and relative price of good j, whereas an increase in demand that rotates the demand curve up ($\xi_{jt}^{\dagger} > 0$) leads to an increase in price with no change in quantity. If the good is demand constrained, productivity shocks $\zeta_{jt}^{\dagger}$ have no effects on either quantity or price, and no new workers will be hired in t, causing employment as a share of the population to fall at the rate of population growth in (77). In addition, aggregate economic conditions also influence output, with a positive value of $n_{1t}^{\dagger}$ leading to lower production of each good j when $\lambda_H < 0$. Aggregate conditions (operating through both $n_{1t}^{\dagger}$ and $X_{1t}^{*\dagger}$) have more complicated general equilibrium implications for relative prices.

Calibration. Our baseline numerical examples use the parameter values in Table 2. We assume that a period corresponds to one quarter, with n implying an annual population growth rate of 1% and β an annual discount rate of 2%. Note that taxes in this model are used solely to finance unemployment compensation, motivating a relatively low value ($\tau = 0.02$) for the marginal tax rate.[7] Productivity for all workers grows at some fixed rate g (which does not affect any of the numbers reported in the table), and the log difference between the most productive and least productive nonspecialized individual ($S_t - R_t$) is constant at 1 for all t. There are huge gross flows out of and into employment in a typical month in the U.S. Davis, Faberman, and Haltiwanger (2006, Table 1) found that 10% of workers lose or quit their jobs each quarter, and the estimates in Ahn and Hamilton (2022) imply that 12% of employed individuals will be unemployed or out of the labor force 3 months later. Our value of $k_X = 0.02$ assumes that involuntary separations account for less than 1/5 of these observed gross flows. When the probability of successfully creating a new good is $k_\pi = 0.25$, the baseline parameters imply a steady-state unemployment rate of $u^0 = 5.1\%$. specialized workers receive 60% of the pretax income of the economy but only account for 55.6% of the population, implying a value for $\omega^0 = 1.0784$. The discounted lifetime advantage of being specialized is $\tilde{V}^0 = 4.80$, which translates into a per-period flow advantage of $[1-\beta(1-k_X)]\tilde{V}^0 = 0.12$, or 12% higher after-tax incomes for specialized workers.

Adjustment dynamics in the absence of new shocks. Adjustment dynamics are straightforward in the case when $Q_{j,t+s} = \bar{Q}_{j,t+s}/2$ and $\chi_{j,t+s} = \xi_{j,t+s} = \zeta_{j,t+s} = 1$ for all $s \geq 0$. In this case equation (64) implies that all specialized workers earn the same income as each other from date t onward

$$y_t = \frac{y^0(1-n_1^0)H_t^2 q_1^0}{(1-n_{1t})\left\{2H_t q_1^0 - n_{1t}\hat{X}_{1t}\right\}}. \tag{79}$$

Here we've introduced the notation $y_t = Y_{jt}/P_{1t}$ to represent the common real income of any

[7] The ratio of the average wage of a nonspecialized worker $P_{1t}^0 \hat{X}_{1t}^0 / h_1^0$ to steady-state unemployment compensation (36) is $[h_1^0 \tau(1-\alpha_1)]/[\alpha_1(1-h_1^0)] = 0.23$.

specialized worker in the absence of shocks and $q_1^0 = Q_{1t}^0/N_t$ for the per capita output of good 1 along the steady-state growth path. We also simplify the notation to the case with zero productivity growth, since $g > 0$ results in a system of equations with identical implications to those presented in this subsection. For example, y^0 in (79) corresponds to $Y_t^0/(1+g)^t$ in the notation that was used in equation (65). With y_t the same for all specialized workers, unemployment compensation in (66) becomes

$$c_t = \frac{\tau(1-n_{1t})y_t}{n_{1t}(1-h_{1t})(1-\tau)} \tag{80}$$

for $c_t = C_t/P_{1t}$. Common incomes also mean that (67) becomes a single value function $\tilde{V}_t$ for all specialties,

$$\tilde{V}_t = \log(y_t) - \log \tilde{X}_{1t} + \beta(1-k_X)\tilde{V}_{t+1} \tag{81}$$

and (68) becomes

$$x_{1t}^* - \log c_t = -k_U + \beta k_\pi \tilde{V}_{t+1} \tag{82}$$

for $x_{1t}^* = \log X_{1t}^*$. The fraction of the population that is nonspecialized evolves as in (71):

$$n_{1,t+1} = 1 - n_{t+1}^\sharp - n_{t+1}^\natural. \tag{83}$$

We assume that new goods enter with $n_{jt} = n_j^0$ when $j \in J_{2t}^\sharp$. However, economic conditions after they enter may cause n_{jt} for $j \in J_{2t}^\natural$ to differ from n_j^0. For example, depressed demand could discourage hiring of new workers, in which case $n_{j,t+s}$ would fall below the value $n_{jt} = n_j^0$ when the good was first introduced. For some of the examples in the next section, it is helpful to keep track of the state variable $\bar{n}_t$ which is defined as the sum of the steady-state employment shares n_j^0 of all specialized goods that are produced at t: $\bar{n}_t = \sum_{j \in \mathcal{J}_{2t}} n_j^0$. A fraction $(1-k_X)$ of goods in t survive to $t+1$, and the value of n_j^0 for these goods at $t+1$ by definition is the same as in t. In addition, for newly produced goods the steady-state population share is the value when they were first introduced: $n_j^0 = n_{jt}$ for $j \in \mathcal{J}_{2t}^\sharp$. Thus the equation of motion for $\bar{n}_t$ is

$$\bar{n}_{t+1} = (1-k_X)\bar{n}_t + n_{t+1}^\sharp. \tag{84}$$

The fraction of the population that is newly specialized is given by (70)

$$n_{t+1}^\sharp = e^{-n}(1-h_{1t})h_{0t}n_{1t}k_\pi. \tag{85}$$

A fraction $(1-k_X)$ of the specialized goods in t survive to $t+1$, and the profit-maximizing employment share for those that survive is $n_{j,t+1}^* = H_{t+1}n_j^0$. Provided $n_{j,t+1}^* \geq n_{jt}$ $\forall j \in J_{2t}$, this means

$$n_{t+1}^\natural = H_{t+1}(1-k_X)\bar{n}_t. \tag{86}$$

Since continuing specialities offer a common income, equilibrium requires in (69) a common probability of successful application π_t:

$$x_{1t}^* - \log c_t = \beta \pi_t \tilde{V}_{t+1}. \tag{87}$$

The value of π_t can be calculated as the ratio of total openings to total applicants:

$$\pi_t = \frac{n_{t+1}^\natural e^n - (1 - k_X)(1 - n_{1t})}{(1 - h_{1t})(1 - h_{0t})n_{1t}}. \tag{88}$$

In addition we have the definitions

$$H_t = 1 + \lambda_H (n_{1t} - n_1^0) \tag{89}$$

$$h_{1t} = \frac{S - x_{1t}^*}{S - R} \tag{90}$$

$$\log \tilde{X}_{1t} = \frac{S^2 - 2Rx_{1t}^* + (x_{1t}^*)^2}{2(S - R)} \tag{91}$$

$$\hat{X}_{1t} = \frac{\exp(S) - \exp(x_{1t}^*)}{S - R}. \tag{92}$$

Equations (79)-(92) are a system of 14 nonlinear dynamic equations in the 14 variables $y_t, c_t, \tilde{V}_t, x_{1t}^*, n_{1t}, \bar{n}_t, n_{t+1}^\sharp, n_{t+1}^\natural, \pi_t, h_{0t}, H_t, h_{1t}, \tilde{X}_{1t}, \hat{X}_{1t}$ in which n_{1t} and $\bar{n}_t$ are predetermined state variables, $\tilde{V}_t$ is a forward-looking state variable, and the other variables can be solved out to arrive at a nonlinear system in the three state variables. Steady-state magnitudes that appear as coefficients in the nonlinear system (for example, y^0, n_1^0, and q_1^0 in (79)) can be obtained by finding the steady-state solution from Proposition 3. For any specified initial n_{1t_0} and $\bar{n}_{t_0}$, the system can be solved using the perfect foresight solver in Dynare (Adjemian et al., 2011) and the terminal conditions $n_{1,t+s} \to n_1^0$, $\bar{n}_{t+s} \to n_1^0$, and $\tilde{V}_{t+s} \to \tilde{V}^0$. Shocks to $\chi_{jt}, \xi_{jt}, \zeta_{jt}$ can be modeled as described in the next two sections. For more details see online Appendix C.

9 Demand shocks.

In this section we consider an economy in which all predetermined variables at t_0 are equal to their steady-state values, meaning $n_{jt} = n_j^0 \ \forall j \in \mathcal{J}_{2t_0}$, $n_{1t_0} = n_1^0$, and $\bar{n}_{t_0} = 1 - n_1^0$. In this section there are no shocks to productivity ($\zeta_{jt} = 1 \ \forall j, t$). In our first two examples, $\chi_{jt} = \chi \neq 1$ for a fraction κ of the specialized goods that were produced at t_0 with $\chi_{jt} = 1$ for the remaining goods and for all $t > t_0$. Note from (64) that nonimpacted goods (those with $\chi_{jt} = 1$) will all offer the same income as each other, which we denote y_t, while income for impacted goods (those with $\chi_{jt} = \chi$) is

$$y_t^{\chi} = \chi^2 y_t. \tag{93}$$

If n_t^{χ} and n_t^c denote the fraction of the population specializing in impacted and nonimpacted goods, unemployment compensation from (66) is given by

$$c_t = \frac{\tau(n_t^c y_t + n_t^{\chi} y_t^{\chi})}{n_{1t}(1 - h_{1t})(1 - \tau)}. \tag{94}$$

9.1 A transient drop in demand.

Let κ denote the fraction of specialized goods that were in production at t_0 for which $\chi_{jt_0} = \chi$. In our first example, 10% of specialized goods experience a 10% drop in demand ($\kappa = 0.1, \chi = 0.9$). Since specialized goods account for 60% of steady-state GDP, this corresponds to a demand shock equal to 0.6% of GDP in t_0. For this example, lower demand lasts for only one period ($\chi_{jt} = 1$ for $t > t_0$).

To determine variables for dates $t > t_0$, note that beginning in $t = t_0+1$, $\chi_{jt} = \xi_{jt} = \zeta_{jt} = 1$ $\forall j$. The profit-maximizing level of production for every specialized good will be $\bar{Q}_{jt}/2 = H_t Q_{jt}^0$ and all teams will add new workers each period to be able to produce this amount. No one has any incentive to give up their specialty in t_0, since $\tilde{V}_{t+1}$ remains quite positive. Thus the economy for $t > t_0$ is described by equations (79)-(92) with the particular path determined by the values of the state variables n_{1,t_0+1} and $\bar{n}_{t_0+1}$ that are endogenously determined at date t_0.

To determine variables at date t_0, note that $n_{1t_0} = n_1^0$, so from (55), $H_{t_0} = 1$. Thus from (54) and (57), impacted goods will produce $\chi Q_{jt_0}^0$ in t_0 while nonimpacted specialized goods produce the steady-state quantities ($Q_{jt_0} = Q_{jt_0}^0$ if $\chi_{jt_0} = 1$). Note that the latter have neither the incentive nor the capacity to produce more than this. The total number of people available to produce good 1 is determined by $n_{1t_0} = n_1^0$, so whether production of good 1 is above or below $Q_{1t_0}^0$ is determined by the productivity threshold x_{1t*}^* above which the nonspecialized produce good 1. Incentives at $t = t_0$ for trying to create a good that will begin production in $t_0 + 1$ are still described by (82). The one variable entering this determination of x_{1t}^* that differs from the steady-state magnitude is unemployment compensation at t_0. This is given by (94), which can be written

$$c_t = \frac{\tau(1 - n_{1t})y_t}{n_{1t}(1 - h_{1t})(1 - \tau)} s_{3t} \tag{95}$$

$$s_{3t} = \begin{cases} 1 + \kappa(\chi^2 - 1) & t = t_0 \\ 1 & t > t_0 \end{cases}.$$

Thus the adjustment path for this example is represented by the system (79)-(92) with (80) replaced by (95). This can again be solved using Dynare with s_{3t} treated as an exogenous shock.

From this solution any other magnitudes of interest can be calculated. For example, for all specialized goods, $n_{jt} = n_j^0$ for $t = t_0$ and $n_{jt} = H_t n_j^0$ for $t > t_0$. This allows us to calculate the relative price of any individual good $p_{jt} = P_{jt}/P_{1t}$ from (59):

$$p_{jt} = \begin{cases} p_j^0 \left[\frac{\chi_{jt_0} n_1^0 \hat{X}_{1t_0}^0}{2n_1^0 \hat{X}_{1t_0}^0 - n_1^0 \hat{X}_{1t_0}} \right] & t = t_0 \\ p_j^0 \left(\frac{1-n_1^0}{1-n_{1t}} \right) \left[\frac{{H_t}^2 n_1^0 \hat{X}_{1t}^0}{2H_t n_1^0 \hat{X}_{1t}^0 - n_{1t} \hat{X}_{1t}} \right] & t = t_0+1, t_0+2, ... \end{cases}.$$

Since $Q_{jt}/(N_t X_{jt}^0) = \chi_{jt} H_t$, real GDP for this example is found from (73) to be

$$Q_t = \begin{cases} (1-\alpha_1)[1+\kappa(\chi-1)] + \alpha_1 \frac{\hat{X}_{1t_0}}{\hat{X}_{1t_0}^0} & t = t_0 \\ \frac{(1-\alpha_1)}{(1-n_1^0)} H_t \bar{n}_t + \frac{\alpha_1 \hat{X}_{1t}}{n_1^0 \hat{X}_{1t}^0} n_{1t} & t > t_0 \end{cases}.$$

The solid green line in Figure 4 plots the path over time of a few selected variables for this example. Real GDP (panel E) falls 0.55% below the steady-state growth path in t_0 but almost completely recovers by $t_0 + 1$. The reason that real GDP does not not fall quite by the full 0.6% expected is because of a modest general-equilibrium feedback. The lower income of the impacted specialized workers in period t_0 results in a decrease in the tax base from which unemployment compensation gets funded. The slight decrease in unemployment compensation induces some nonspecialized individuals in t_0 to produce good 1 rather than train for a specialty (panel C). The increase in production of good 1 slightly offsets the lost production of some specialized goods, explaining why GDP falls by only 0.55% rather than 0.6%.

The drop in demand for impacted goods leads to an increase in the relative price of all other goods. But this increase in relative prices results in little or no increase in production of those goods because the productive resources that become underutilized in the sectors with lower demand cannot be costlessly reallocated to produce in other sectors. Since the drop in demand only lasts one period, there is very little reallocation over time either. To a first approximation, a transient drop in demand results in a transient drop in real GDP from which the economy almost immediately recovers.

9.2 A transient increase in demand.

Consider next the case in which there is a transitory 10% increase in demand ($\chi = 1.1$) affecting 10% of the specialized goods. Since these goods would have been at capacity with $\chi = 1$, their response at t_0 is to increase price with no change in production. The time paths in this case are plotted in dashed blue in Figure 4. In this case tax receipts (in units of good 1) go up, providing more generous unemployment compensation that leads to a slight reduction of good 1. The result is that real GDP actually falls very slightly in response to an increase in demand for goods that are already capacity constrained. To summarize: a reduction in demand will reduce real GDP, but an increase in demand need not increase real GDP.

9.3 A large persistent but isolated drop in demand.

Next consider the case of a 40% drop in demand that affects only 2.5% of specialized goods ($\chi = 0.6$, $\kappa = 0.025$). Note that the total size of the shock to demand is the same as in Example 9.1 (with $\kappa(\chi - 1) = -0.01$ in both cases) but in Example 9.3 the drop in demand is concentrated on a small subset of goods. If the low demand only lasted for a single period, the results would be identical to those in Example 9.1. Here however we consider a shock that lasts for $D = 8$ periods. We take the shock to be isolated in the sense that new goods created beginning in $t_0 + 1$ all enjoy the steady-state demand level $\chi_{jt} = 1$. From (64) the period t log income for workers specializing in the impacted good would differ from that of other specialized workers by $\log \chi^2$ for each $t = t_0, ..., t_0 + D - 1$. From (67) this means that the lifetime advantage as of date t of having a specialty in the impacted good is

$$\tilde{V}_t^{\chi} = \tilde{V}_t + \beta_{t_0+D-t}^X \log \chi^2 \tag{96}$$

$$\beta_{t_0+D-t}^X = \begin{cases} \sum_{s=0}^{t_0+D-t-1} [\beta(1 - k_X)]^s & t = t_0, ..., t_0 + D - 1 \\ 0 & t \geq t_0 + D \end{cases} . \tag{97}$$

If the drop in demand is big enough and lasts long enough, it could turn out that $\tilde{V}_{t_0+1}^{\chi} < 0$ which would mean $E_{t_0} V_{j,t_0+1}$ for impacted goods is less than $E_{t_0} V_{1,t_0+1}$. In this case individuals who had specialized in the impacted goods would be better off abandoning their specialty and returning to the pool of the nonspecialized and the possibility of developing a new specialty. This turns out to be the case for the size of the shock in this numerical example. All impacted goods produce $\chi Q_{jt_0}^0$ in period t_0 and are then discontinued.

The effects of discontinued goods can be represented as shocks to the equations that determine $n_{t+1}^{\natural}$ and $\bar{n}_{t+1}$ by generalizing (86) and (84) to

$$n_{t+1}^{\natural} = H_{t+1}(1 - k_X) s_{7t} \bar{n}_t$$

$$\bar{n}_{t+1} = (1 - k_X)\bar{n}_t s_{1t} + n_{t+1}^{\sharp}.$$

For Example 9.3, $s_{1t} = s_{7t} = (1 - \kappa)$ for $t = t_0$ and equal to unity afterwards. Finally, the probability in (88) of successfully obtaining a job with a continuing specialty is modified to reflect the fact that discontinued goods will not be hiring in $t_0 + 1$:

$$\pi_t = \frac{n_{t+1}^{\natural} e^n - (1 - k_X)(1 - n_{1t}) s_{6t}}{(1 - h_{1t})(1 - h_{0t}) n_{1t}}$$

with $s_{6t} = (1-\kappa)$ for $t = t_0$ and equal to unity afterwards.

The time paths for variables in this example are shown as solid black lines in Figure 4. The fact that there are fewer specialists in t_0+1 raises the value of $\tilde{V}_{t_0+1}$ (panel B), and this partially offsets the effect of χ on $\tilde{V}^{\chi}_{t_0+1}$ in equation (96). However, for this numerical example it is the case that all impacted specialists still want to return to the pool of nonspecialized despite the higher value of $\tilde{V}_{t_0+1}$ (see panel D)[8] With the higher anticipated rewards to specialization, more nonspecialized than usual start trying to develop a specialty in t_0 (panel C). This results in a lower level of production of good 1 in t_0 that contributes to the GDP loss resulting from lower production of specialized goods (panel E).

Demand for all new and surviving goods is back to normal beginning in t_0+1. But GDP is not back to steady state because of the surplus of nonspecialized individuals. Why doesn't the economy return to steady state immediately? The distribution of productivity of nonspecialized workers x_{it} results in a continuum of opportunity costs across nonspecialized workers who might consider trying to develop a specialization. At any point in time, some nonspecialized individuals find it worth their while to search for something better and others do not. More than the usual number will be searching as long as $n_{1t} > n_1^0$, and this will eventually return the economy to $n_{1t} = n_1^0$. But the effects of a demand shock can persist long after the shock is gone if it takes time for the overall population to develop new specializations.

This example offers one possible interpretation of why goods are always being discontinued along the steady-state growth path. Suppose that along the steady-state growth path, each period t a fraction k_X of specialized producers learn that demand for their particular product is going to experience a decrease of the magnitude considered in this example beginning in $t+1$. Producers of those goods would have an incentive to abandon their specialty after producing in t, and choose to return to the pool of nonspecialized workers. Thus a shock of the kind considered in Example 9.3 could be viewed as occurring all the time in this model. Everyone takes into account the possibility that the good could be discontinued at any time (and eventually will be discontinued for certain) through the parameter k_X that enters every decision. Example 9.3 could be viewed as exploring what happens when these regular demand shocks affect a larger fraction of goods than usual and catch the producers of these goods by surprise in t_0.

9.4 The role of technological frictions.

How long it takes to return to steady state depends on how hard it is to create new goods. Here we illustrate this with an example that is identical to Example 9.3 except that now the

[8] For a more modest shock χ, it could be the case that if all impacted goods were to drop out, the value of $\tilde{V}_{t_0+1}$ would be sufficiently big that an impacted specialist would want to remain in, whereas if they all remained in then $\tilde{V}_{t_0+1}$ would be sufficiently small that everyone would want to drop out. In this case the equilibrium would be characterized by a fraction κ_χ of impacted goods being discontinued such that $\tilde{V}^{\chi}_{t_0+1}$ exactly equals zero, i.e., those who remain are just indifferent between staying or leaving.

probability of successfully creating a new specialized good is $k_\pi = 0.60$ rather than 0.25 in our baseline parameterization. With lower technological frictions to developing new goods, there is a lower equilibrium unemployment rate ($u^0 = 2.4\%$ versus 5.1% for the baseline parameters) and a lower equilibrium advantage to being specialized ($\tilde{V}^0 = 0.7$ versus 4.8 for the baseline case). If this economy were subjected to the same shock as in Example 9.3, all impacted specialists would again choose to return to the pool of the nonspecialized. Indeed, they would have chosen to do so with a much smaller or less persistent shock, since a new specialization is much more easily obtained when $k_\pi = 0.6$. This is reflected in the very negative value for the red plot of $\tilde{V}_t^\chi$ in panel D of Figure 4. Although the surge in nonspecialized workers in $t_0 + 1$ is the same as in Example 9.3, the economy recovers more quickly.

9.5 A persistent drop in demand for new and existing goods.

The assumption in Examples 9.3 and 9.4 was that newly created goods were immune from the lower demand that hit existing goods. In those examples, a surge in new good creation was a key factor that helped the economy recover. In reality, starting a new business may be harder than usual when the economy is weak. To study this possibility, we now consider a demand shock that affects not just goods that were produced at t_0 but also any new goods that are introduced during the period when demand remains weak. Just as in Example 9.1, we suppose that 10% of existing goods at t_0 experience a 10% drop in demand ($\kappa = 0.1$, $\chi = 0.9$), but now we assume that low demand persists for $D = 5$ periods. For these numerical values, it turns out that impacted specialized workers would want to retain their specialty and wait until demand recovers. But impacted goods are demand constrained and do no additional hiring until demand recovers. For the first $D - 1$ periods the condition for new good creation (82) becomes

$$x_{1t}^* - \log c_t = -k_U + \beta k_\pi \tilde{V}_{t+1} + \beta k_\pi \beta_{t_0+D-t-1}^X \log \chi^2$$

for β_j^X given by (97). Some nonspecialized workers will still choose this option, though fewer do so than would in steady state because of the lower return to trying to develop a specialization.

Let J_{2t}^χ denote the set of goods produced at t for which $\chi_{jt} = \chi$. This set consists of those goods that survive to t that either experienced $\chi_{jt_0} = \chi$ in t_0 or were newly created between t_0+1 and t_0+D-1. For this example we keep track of the fraction of the population with impacted specialties, which evolves during the period of depressed demand according to $n_{t+1}^\chi = e^{-n}(1-k_X)n_t^\chi + n_{t+1}^\sharp$, and the sum of n_j^0 for all impacted goods, $\bar{n}_t^\chi = \sum_{j \in J_{2t}^\chi} n_j^0$. The latter evolves according to $\bar{n}_{t+1}^\chi = (1-k_X)n_t^\chi + n_{t+1}^\sharp$ since new goods enter with $n_j^0 = n_{jt}$ for $t \in J_{2t}^\sharp$ and continuing goods retain n_j^0 as long as they continue to be produced. We also have $\bar{n}_t^c = \sum_{j \in J_{2t}^c} n_j^0$ whose value is given by $\bar{n}_t^c = (1-k_X)^{t-t_0}(1-\kappa)(1-n_1^0)$. These changes can again be represented by a series of shocks for periods $t_0, ..., t_0 + D - 1$ after which the system dynamics revert to (79)-(92). For details see online Appendix C.

The cyan curves in Figure 4 show the adjustment dynamics for this example. The lower rewards to specializing (panel D) and the reduced hiring opportunities from continuing goods result in a larger fraction of the nonspecialized deciding to produce good 1 (panel C). This higher production of good 1 helps offset much of the initial loss in GDP in t_0 that resulted from less production of impacted specialized goods. But since fewer of the nonspecialized are investing in specialization, we see in panel A an increase over time in the fraction of the population that are nonspecialized. This continues to drag GDP down as long as the period of weak hiring persists, and by period $t_0 + 4$ GDP is 1% below trend. Demand conditions are fully recovered beginning in $t_0 + 5$, but the economy only gradually returns to steady state for the same reasons as in Example 9.3. This provides an illustration of the possible hump-shaped response to a demand shock mentioned in the introduction. In this example, the normal inflow of people who are looking for better jobs confronts a slower than normal rate of new hiring. The effects of this on the number of nonspecialized workers and the value of GDP build over time.

9.6 Shocks to ξ_{jt}.

Up to this point we have been discussing shocks to the preference parameter χ_{jt}, which results in a vertical shift of the demand curve for good j and changes the profit-maximizing level of output $\bar{Q}_{jt}/2$ (see Figure 1). Consider now a shock to the parameter ξ_{jt}, which changes the vertical intercept A_{jt} of the demand curve but leaves the horizontal intercept $\bar{Q}_{jt}$ and the profit-maximizing level of output $\bar{Q}_{jt}/2$ unchanged. From equation (57), this has no direct effect on output regardless of whether the good is demand- or supply-constrained. It results instead in an increase in the relative price of good j and in the relative income of producers of good j. The changes in income will result in a change in tax receipts and unemployment compensation which would have general-equilibrium effects on h_{1t} and n_{1t}, but these would be secondary contributions of the size noted in Example 9.1.

It is possible that if a drop in ξ_{jt} is large enough and lasts long enough, the drop in income for producers of the good would be sufficiently large to persuade them to discontinue production, but we do not explore this possibility here.

Although shocks to ξ_{jt} are less interesting for purposes of the questions studied here than are shocks to χ_{jt}, it would be important to include them in any empirical applications. The three shocks $\chi_{jt}, \xi_{jt}, \zeta_{jt}$ could be viewed as the fundamental drivers (along with aggregate factors represented here by n_{1t} and x_{1t}^*) of the three variables Q_{jt}, p_{jt}, N_{jt}. If we think of observed output for a given sector or firm k as resulting from a collection of $J_t^{[k]}$ separate production activities (e.g., $N_t^{[k]} = \sum_{j \in J_t^{[k]}} N_{jt}$), some fraction of which could be demand-constrained and others supply-constrained, the three observed magnitudes $Q_t^{[k]}, p_t^{[k]}, N_t^{[k]}$ could be interpreted as driven by a mixture of three unobserved shocks $\chi_t^{[k]}, \xi_t^{[k]}, \zeta_t^{[k]}$.

10 Productivity shocks.

In this section we consider an economy that begins period t_0 with all predetermined variables equal to their steady-state values and $\chi_{jt} = \xi_{jt} = 1 \ \forall j, t$. We study the consequences if the productivity parameter $\zeta_{jt} = \zeta \neq 1$ for a fraction κ of the specialized goods in production in date t_0.

10.1 A transient drop in productivity.

Consider first a 10% drop in productivity that affects 10% of the specialized goods at t_0 ($\zeta = 0.9$, $\kappa = 0.1$) with productivity returning to normal in t_0+1. Since $n_{1t_0} = n_1^0$ and $H_{t_0} = 1$, from (57) nonimpacted goods will produce $Q^0_{jt_0}$ while impacted goods will produce $\zeta Q^0_{jt_0}$. Moreover, $\bar{Q}_{jt_0} - Q_{jt_0} = (2-\zeta)Q^0_{jt_0} = 1.1Q^0_{jt_0}$ for $j \in \mathcal{J}^{\zeta}_{2t_0}$ so from (58) impacted goods raise their price by 10% relative to nonimpacted goods. With a 10% drop in production and a 10% increase in price, the effect on relative income is nearly a wash; $Q_{jt_0}(\bar{Q}_{jt_0} - Q_{jt_0}) = \zeta(2-\zeta)(Q^0_{jt_0})^2$ so from (63) $y^{\zeta}_t = \zeta(2-\zeta)y_t$. For $\zeta = 0.9$, $\zeta(2-\zeta) = 0.99$, implying a 1% drop in the relative income of impacted workers. Thus the general equilibrium effects operating through unemployment compensation c_{t_0} are even more modest here than in Example 9.1. We would see nearly a 0.6% drop in real GDP at t_0 followed by an almost complete recovery in t_0+1, with the time paths of variables in Example 10.1 almost the same as the solid green lines in panels A-C and E of Figure 4.

The effects of supply and demand shocks could look quite similar to each other in the data. Note moreover that we could not use measured productivity to distinguish between the 10% demand shock in Example 9.1 and the 10% productivity shock in Example 10.1. In both cases, output falls by 10% with no change in labor input, so measured productivity of impacted goods falls by 10% in both examples. The one way we could distinguish between demand and supply shocks is in observations of relative prices. In Example 9.1 the drop in production is accompanied by a decrease in the relative price, whereas in Example 10.1 we would see an increase in the relative price.

10.2 A transient increase in productivity.

Consider next the case in which a fraction $\kappa = 0.1$ of the goods in production at t_0 experience a 10% increase in productivity ($\zeta = 1.1$), with conditions again returning to normal beginning in t_0+1. Although more goods could be produced in t_0, no one has an incentive to do so, since $\bar{Q}_{jt_0}/2$ is still the profit-maximizing level of production. There is no incentive to change prices, and no incentive to make any changes for the future since conditions at t_0+1 will be back to steady state. If businesses already have the capacity to produce at the revenue-maximizing level of output, a purely transient increase in productivity has no effect on the output or price of any good at any date.

10.3 A persistent decrease in productivity.

We saw in Example 9.3 that if a drop in demand is sufficiently severe and long lasting it could induce workers to abandon their specialty. How big a drop in productivity would be necessary to produce the same result? A demand shock lowers income by a factor of χ^2 whereas the factor for a productivity shock is $\zeta(2-\zeta)$. For $\chi = \zeta = 0.9$, $\chi^2 = 0.81$ and $\zeta(2-\zeta) = 0.99$. A 10% drop in demand would have a significant effect on income, whereas the effect on income of a 10% drop in productivity would be negligible. In Example 9.3 we saw that a 40% drop in demand ($\chi = 0.6$) that persisted for two years would be sufficient to persuade workers to try to develop a specialization. To get a comparable income effect from a productivity shock would require an 80% drop in productivity that persists for two years.[9] Thus in this model both demand and supply shocks could contribute to short-run economic fluctuations, but it would require quantitatively bigger productivity shocks to produce some of the propagation mechanisms to which we have called attention.

11 Discussion.

Although labor is the only factor of production in this model, the mechanisms explored here apply more generally to production that relies on a network of interdependent specialized resources. And although we model each production team as producing a single, stand-alone final good, our view is that most firms in the economy make use of a number of separate teams to produce a variety of differentiated products.

In order to focus as clearly as possible on the role of specialization in determining the level of economic activity, this paper abstracted from many details that play a key role in economic fluctuations. Here labor was the only input, with specialization taking the form of training and assembling a dedicated team of workers. Specialized capital is an even more important commitment for most businesses (Ramey and Shapiro, 1998 and 2001). Production moreover typically depends on inputs purchased from other firms that themselves specialize to be able to provide those goods or services, amplifying the forces studied here through network connections; see Baqaee, 2018, Baqaee and Farhi, 2019, and Bigio 2021. This paper completely ignored financial frictions, even though they appear to be a key factor in many economic downturns. And although nominal frictions played no role in this model, they could well be an additional factor in amplifying economic downturns, just as they are known to amplify the effects of productivity shocks in existing models that incorporate costs of reallocating resources across sectors (e.g., Guerrieri et al., 2020).

By focusing on just a single source of specialization and a single technological friction, the hope was to shed light on the interaction between specialization and demand as a fundamental short-run determinant of the level of GDP.

[9] For $\chi = 0.6$ and $\zeta = 0.2$, $\chi^2 = \zeta(2-\zeta) = 0.36$.

12 Conclusion.

There is a consensus among many macroeconomists that demand shocks are important in the short run but not in the long run. The usual explanation for why there is a difference between the short run and the long run is the claim that wages and prices are slow to adjust. This paper emphasized a different channel: productive resources take time to reallocate.

Both demand shocks and productivity shocks could contribute to short-run fluctuations in the model here. Why did I emphasize the role of demand? Consumer preferences change all the time. But events that lead to significant drops in productivity are harder to identify as the cause of historical economic downturns. The popularity of macroeconomic models in which productivity shocks are the primary cause of short-run economic fluctuations is based not on anything observed in the data but instead on a lack of satisfactory models with which we could understand how demand shocks could be the main cause of business cycles. This is why I wrote this paper.

References.

Adjemian, Stéphane et al. (2011), "Dynare: Reference Manual, Version 4," Dynare Working Papers, 1, CEPREMAP.

Ahn, Hie Joo, and James D. Hamilton (2022). "Measuring Labor-Force Participation and the Incidence and Duration of Unemployment," *Review of Economic Dynamics* (4): 1-32.

Alvarez, Fernando, and Robert Shimer (2011). "Search and Rest Unemployment," *Econometrica* 79: 75-122.

Angeletos, George-Marios (2018). "Frictional Coordination," *Journal of the European Economic Association* 16: 563-603.

Angeletos, George-Marios, and Chen Lian (2020). "Confidence and the Propagation of Demand Shocks," working paper, MIT.

Auclert, Adrien, Matthew Rognlie, and Ludwig Straub (2020). "Micro Jumps, Macro Humps: Monetary Policy and Business Cycles in an Estimated HANK Model," working paper.

Auerbach, Alan J., Yuriy Gorodnichenko, and Daniel Murphy (2020). "Macroeconomic Frameworks: Reconciling Evidence and Model Predictions from Demand Shocks," NBER Working Paper 26365.

Baqaee, David Rezza (2018). "Cascading Failures in Production Networks," *Econometrica* 86: 1819-1838.

Baqaee, David Rezza, and Emmanuel Farhi (2019). "The Macroeconomic Impact of Microeconomic Shocks: Beyond Hulten's Theorem," *Econometrica* 87: 1155-1203.

Ball, Laurence, and N. Gregory Mankiw (1994). "Asymmetric Price Adjustment and Economic Fluctuations," *Economic Journal* 104: 247-261.

Bigio, Saki (2021). "A Theory of Payments-Chain Crises," working paper, UCLA.

Christiano, Lawrence J., Martin Eichenbaum, and Charles L. Evans (2005). "Nominal Rigidities and the Dynamic Effects of a Shock to Monetary Policy," *Journal of Political Economy* 113: 1-45.

Cooper, Russell, and Andrew John (1988). "Coordinating Coordination Failures in Keynesian Models," *Quarterly Journal of Economics* 103: 441-463.

Davis, Steven J., R. Jason Faberman, and John Haltiwanger (2006). "The Flow Approach to Labor Markets: New Data Sources and Micro-Macro Links," *Journal of Economic Perspectives* 20: 3-26.

Fajgelbaum, Pablo D., Edouard Schaal, and Mathieu Taschereau-Dumouchel (2017). "Uncertainty Traps," *Quarterly Journal of Economics* 132: 1641-1692.

Guerrieri, Veronica et al. (2020). "Macroeconomic Implications of COVID-19: Can Negative Supply Shocks Cause Demand Shortages?", NBER Working Paper No. 26918.

Guerrieri, Veronica et al. (2021). "Monetary Policy in Times of Structural Reallocation," in *Proceedings of the 2021 Jackson Hole Symposium*, pp. 17-52, Federal Reserve Bank of Kansas City.

Hamilton, James D. (1988). "A Neoclassical Model of Unemployment and the Business Cycle," *Journal of Political Economy* 96: 593-617.

Hamilton, James D. (2008). "Daily Monetary Policy Shocks and New Home Sales," *Journal of Monetary Economics* 55: 1171-1190.

Ilut, Cosmin, and Hikaru Saijo (2021). "Learning, Confidence, and Business Cycles," *Journal of Monetary Economics* 117: 354-376.

Kaas, Leo, and Philipp Kircher (2015). "Efficient Firm Dynamics in a Frictional Labor Market," *American Economic Review* 105: 3030-60.

Kaplan, Greg, and Guido Menzio (2016). "Shopping Externalities and Self-Fulfilling Unemployment Fluctuations," *Journal of Political Economy* 124: 771-825.

Lo, Ming Chien, and Jeremy Piger (2005). "Is the Response of Output to Monetary Policy Asymmetric? Evidence from a Regime-switching Coefficients Model," *Journal of Money, Credit and Banking*: 865-886.

Martellini, Paolo, and Guido Menzio (2020). "Declining Search Frictions, Unemployment, and Growth," *Journal of Political Economy* 128: 4387-4437.

Mian, Atif, and Amir Sufi (2014). "What Explains the 2007–2009 Drop in Employment?" *Econometrica* 82: 2197-2223.

Michaillat, Pascal, and Emmanuel Saez (2015). "Aggregate Demand, Idle Time, and Unemployment," *Quarterly Journal of Economics* 130: 507-569.

Murphy, Daniel (2017). "Excess Capacity in a Fixed-Cost Economy," *European Economic Review* 91: 245-260.

Ramey, Valerie A., and Matthew D. Shapiro (1998). "Costly Capital Reallocation and the Effects of Government Spending," *Carnegie-Rochester Conference Series on Public Policy* 48:

145-194.

Ramey, Valerie A., and Matthew D. Shapiro (2001). "Displaced Capital: A Study of Aerospace Plant Closings," *Journal of Political Economy* 109: 958-992.

Şahin, Ayşegül, et al. (2014). "Mismatch Unemployment," *American Economic Review* 104: 3529-3564.

Tobin, James (1972). "Inflation and Unemployment," *American Economic Review* 62:1-18.

Weise, Charles L. (1999). "The Asymmetric Effects of Monetary Policy: A Nonlinear Vector Autoregression Approach," *Journal of Money, Credit and Banking*: 85-108.

Woodford, Michael (1991). "Self-Fulfilling Expectations and Fluctuations in Aggregate Demand," in *New Keynesian Economics, Volume 2, Coordination Failures and Real Rigidities,* edited by N. Gregory Mankiw and David Romer, MIT Press.

Table 1. Ranges and derivatives of key variables.

	Variable	Value at	Value at	Sign of	Steady-state
	Z_t	$\log X_{1t}^* = R_t$	$\log X_{1t}^* = S_t$	$\partial Z_t / \partial \log X_{1t}^*$	derivative
(1)	$\log X_{1t}^*$	R_t	S_t	> 0	
(2)	$\log(1 - h_{1t})$	$-\infty$	0	> 0	$\lambda_2 = \frac{1}{\log X_{1t}^{*0} - R_t}$
(3)	$\log \tilde{X}_{1t}$	$\frac{S_t + R_t}{2}$	S_t	> 0	$\lambda_3 = \frac{\log X_{1t}^{*0} - R_t}{S_t - R_t}$
(4)	$\hat{X}_{1t}$	$\frac{\exp(S_t) - \exp(R_t)}{S_t - R_t}$	0	< 0	
(5)	$\log \hat{X}_{1t}$	$> \frac{S_t + R_t}{2}$	$-\infty$	< 0	$\lambda_5 = \frac{-X_{1t}^{*0}}{\hat{X}_{1t}^0 (S_t - R_t)}$
(6)	h_{Yt}	$-\infty$	∞	> 0	
(7)	$\tilde{V}_t$	finite	$-\infty$	< 0	

Table 2. Parameter values used in baseline calculations.

Exogenous parameters

parameter	**meaning**
$\alpha_1 = 0.4$	steady-state expenditure share of good 1
$\tau = 0.02$	marginal tax rate
$\beta = 0.995$	discount rate
$k_U = 0.2$	utility cost of trying to create new good
$k_\pi = 0.25$	probability of successfully creating new good
$k_X = 0.02$	fraction of goods discontinued each period in steady state
$n = 0.0025$	population growth rate
$R_{t_0} = 1$	initial lowest log productivity of nonspecialized workers
$S_{t_0} = 2$	initial highest log productivity of nonspecialized workers

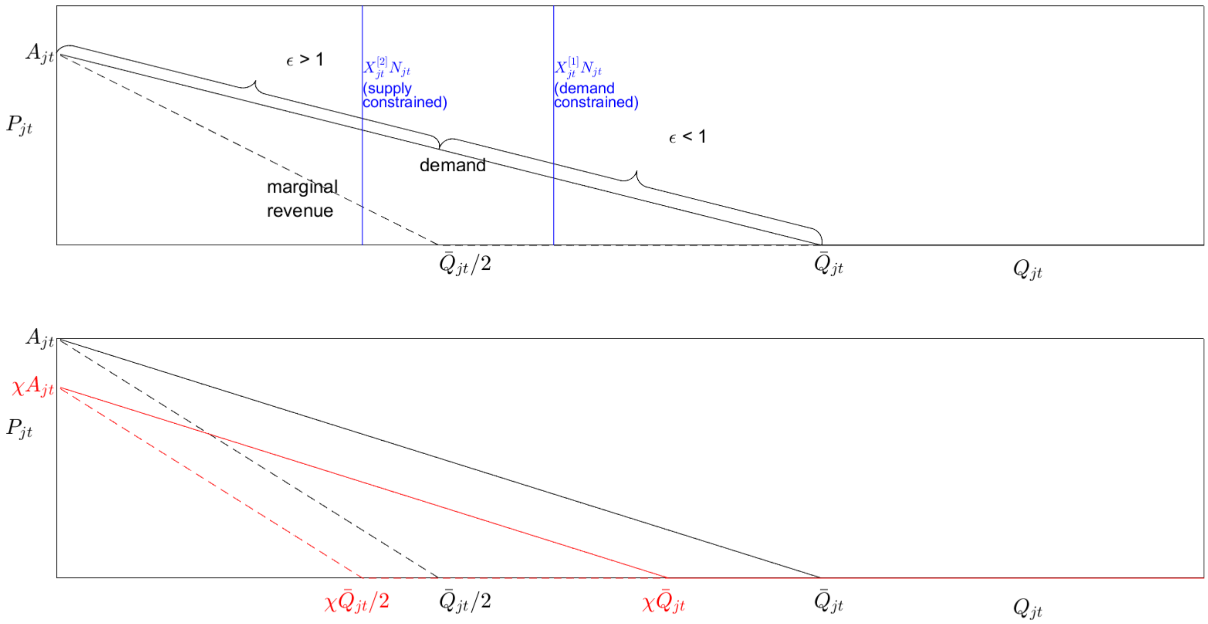
Derived parameters

parameter	**meaning**
$\lambda_2 = 8.668$	elasticity of nonspecialized unemployment $(1 - h_{1t})$ with respect to threshold X_{1t}^*
$\lambda_3 = 0.1154$	elasticity of flow-value of nonspecialized $\tilde{X}_{1t}$ with respect to threshold X_{1t}^*
$\lambda_5 = -0.7032$	elasticity of productivity of unemployed $\hat{X}_{1t}$ with respect to threshold X_{1t}^*
$\lambda_H = -0.1281$	semi-elasticity of demand parameter $\bar{Q}_{jt}$ with respect to fraction of nonspecialized n_{1t}

Steady-state values of endogenous variables

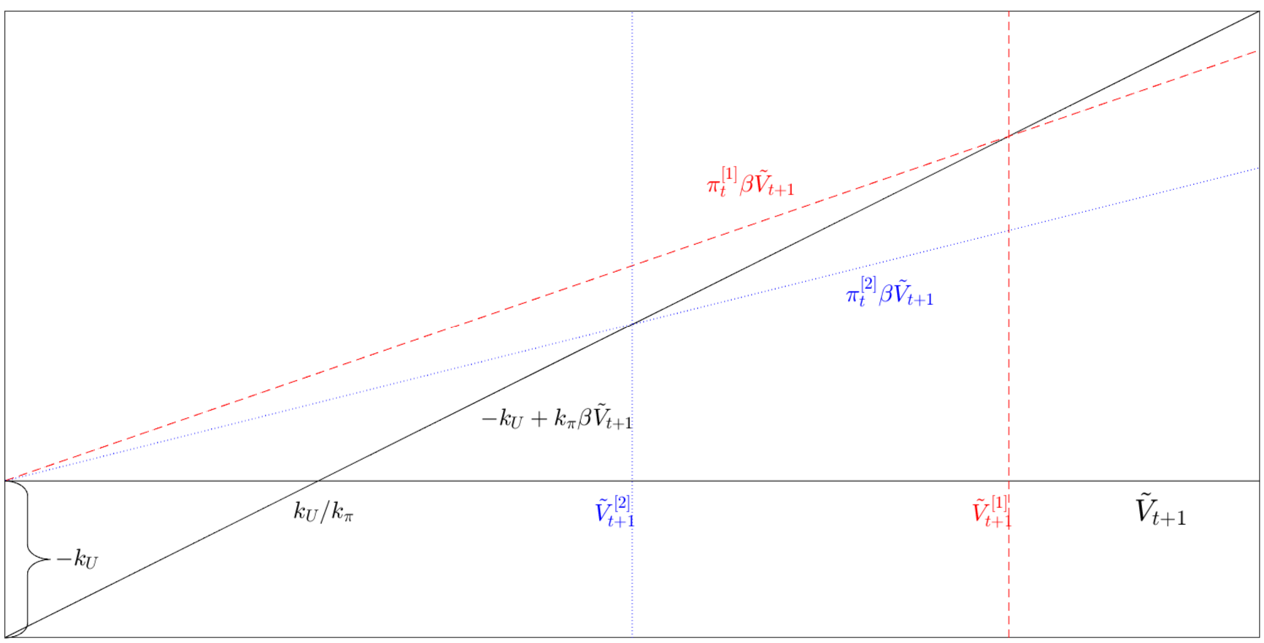
variable	**meaning**
$n_1^0 = 0.4436$	fraction of population without a specialization
$\log X_{1t_0}^{*0} = 1.1154$	initial productivity threshold for nonspecialized workers to produce good 1
$1 - h_1^0 = 0.1154$	fraction of nonspecialized workers who are unemployed
$h_0^0 = 0.8719$	fraction of nonspecialized who try to create new goods
$u^0 = 0.0512$	fraction of population who are unemployed
$\pi^0 = 0.2082$	probability of successfully becoming specialized in an existing good
$\omega^0 = 1.0784$	ratio of spending to employment share of specialized goods
$\tilde{V}^0 = 4.8032$	discounted lifetime log income differential between specialized and nonspecialized

Figure 1. Market demand and marginal revenue.



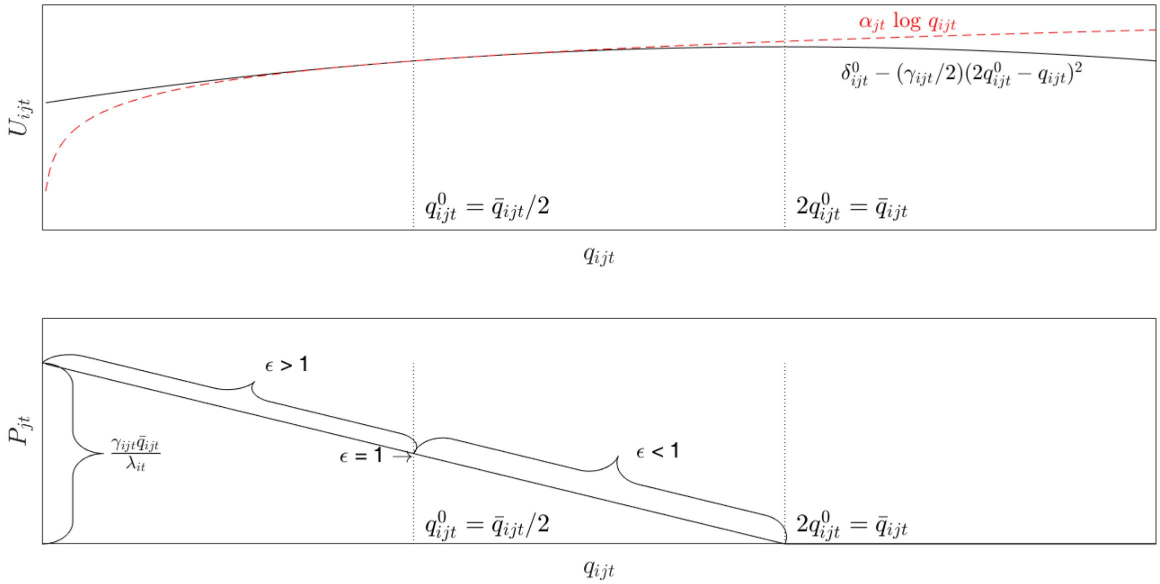
Notes to Figure 1. Top panel: demand, elasticity of demand, and marginal revenue for good *j*. Bottom panel: effects on demand and marginal revenue if the marginal utility of good *j* for all consumers decreases by a factor χ .

Figure 2. Benefit of creating new good versus specializing in existing good.



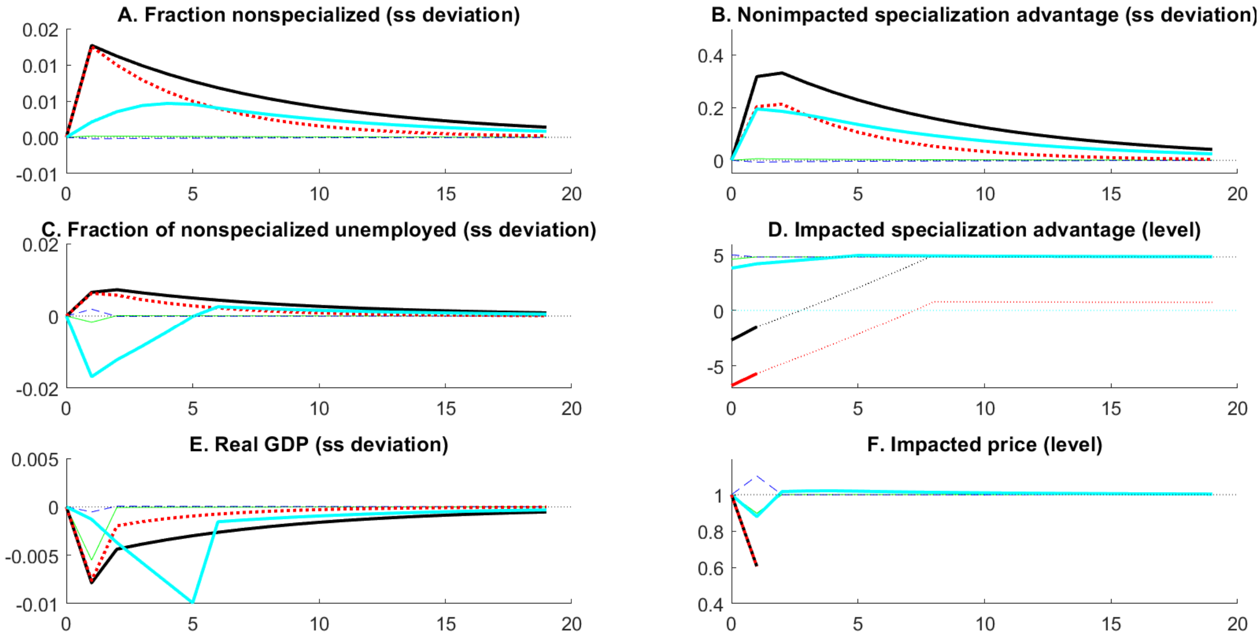
Notes to Figure 2. Horizontal axis: advantage of specialization ($\tilde{V}_{t+1}$). Vertical axis: benefit to trying to create new good (solid black) and of specializing in existing good for two different values of π_t (dashed red and dotted blue).

Figure 3. Individual utility and demand curves.



Notes to Figure 3. Top panel: logarithmic preferences and quadratic approximation. Bottom panel: demand curve associated with quadratic preferences.

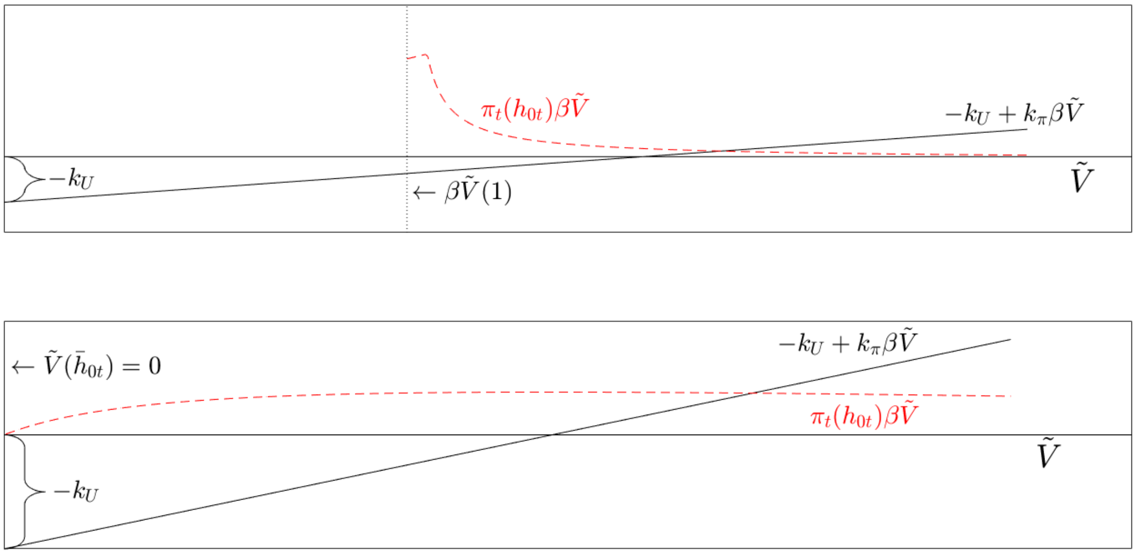
Figure 4. Effects of demand shocks.



Notes to Figure 4. Horizontal axis: time ($t_0 = 1$). Vertical axis: deviation of variable from value on the steady-state growth path for Panels A-C and E; levels in Panels D and F. Panel A: fraction of population without a specialization ($n_{1t} - n_1^0$). Panel B: lifetime advantage of specializing in goods that do not experience demand shock ($\tilde{V}_t - \tilde{V}^0$). Panel C: fraction of nonspecialized workers who are unemployed ($-h_{1t} + h_1^0$). Panel D: lifetime advantage of specializing in goods that experience demand shock ($\tilde{V}_t^{\chi}$). Panel E: log of real GDP ($logQ_t$). A value of -0.01 on the vertical axis represents a value of real GDP that is 1% below the value on the steady-state growth path. Panel F: relative price that would maximize profits for goods that experience demand shocks (p_{jt}^{χ}) with steady-state relative price normalized at 1. Solid green (Example 9.1): 10% of specialized goods experience 10% drop in demand that only lasts for period t_0. Dashed blue (Example 9.2): 10% of specialized goods experience 10% increase in demand that only lasts for period t_0. Solid black (Example 9.3): 2.5% of specialized goods experience 40% drop in demand that would last for 8 periods if goods remained in production. Dotted red (Example 9.4): same as Example 9.3 except that creating a new good is easier ($k_\pi = 0.6$). Solid cyan (Example 9.5): 10% of specialized goods and all newly created goods experience a 10% drop in demand that lasts for 5 periods.

.

Figure A1. Value of specialization solved in terms of h_{0t} and $X_{1t}^*(h_{0t})$.



Notes to Figure A1. Each value of h_{0t} implies a steady-state fraction of the population without a specialization and thus particular values of $X_{1t}^*(h_{0t})$ and $\tilde{V}(h_{0t}, X_{1t}^*(h_{0t}))$. Each point on the horizontal axis corresponds to a particular value of h_{0t} and its implied $X_{1t}^*(h_{0t})$ and $\tilde{V}(h_{0t}, X_{1t}^*(h_{0t}))$ with that value of $\tilde{V}$ plotted on the horizontal axis. Thus h_{0t} is decreasing and X_{1t}^* increasing as we move to the right along the horizontal axis. The vertical axis plots the value of trying to create a new good (in black) or seeking to specialize in an existing good (in dashed red) as a function of that $\tilde{V}(h_{0t}, X_{1t}^*(h_{0t}))$. The two panels correspond to different parameter configurations depending on whether $\tilde{V}(1, X_{1t}^*(1))$ is positive (top panel) or nonpositive (bottom panel).

Appendix A. Proofs of propositions

Proof of Proposition 1.

(a) If $q_{ijt} = \bar{q}_{ijt}/2$, equation (3) becomes

$$\gamma_{ijt}\bar{q}_{ijt}/2 = \lambda_{it}P_{jt}. \tag{A1}$$

Multiplying both sides of (A1) by $\bar{q}_{ijt}/2 = q_{ijt}$ gives

$$\alpha_{ijt} = \lambda_{it}P_{jt}q_{ijt}. \tag{A2}$$

Summing (A2) over j and using $\sum_{j\in J_t} P_{jt}q_{ijt} = y_{it}$ gives

$$\textstyle\sum_{j\in J_t}\alpha_{jt} = \lambda_{it}y_{it}. \tag{A3}$$

Dividing (A2) by (A3) gives (6).

(b) For all $i \in M_t^{(k)}$ we have from (3) that

$$\gamma_{jt}^{(k)}(\bar{q}_{jt}^{(k)} - q_{ijt}) = \lambda_{it}P_{jt}.$$

Integrating over $i \in M_t^{(k)}$, dividing by $R_t^{(k)}$, and using (7) gives

$$\gamma_{jt}^{(k)}(\bar{q}_{jt}^{(k)} - \bar{q}_{jt}^{(k)}/2) = P_{jt}\lambda_t^{(k)} \tag{A4}$$

for $\lambda_t^{(k)} = (1/R_t^{(k)})\int_{i\in M_t^{(k)}}\lambda_{it}di$. Multiplying (A4) by (7) gives

$$\alpha_{jt}^{(k)} = \frac{\lambda_t^{(k)}P_{jt}\int_{i\in M_t^{(k)}} q_{ijt}di}{R_t^{(k)}}. \tag{A5}$$

Summing over j gives

$$\textstyle\sum_{j\in J_t}\alpha_{jt}^{(k)} = \frac{\lambda_t^{(k)}}{R_t^{(k)}}\int_{i\in M_t^{(k)}} y_{it}di. \tag{A6}$$

Dividing (A5) by (A6) gives (8).

Proof of Proposition 2.

Let $z \sim U(R, S)$: $f(z) = (S-R)^{-1}$ for $z \in [R, S]$. Then:

(a)

$$P(z \geq z^*) = \int_{z^*}^{S} \frac{1}{S-R}dz = \left.\frac{z}{S-R}\right|_{z^*}^{S} = \frac{S-z^*}{S-R};$$

(b)

$$\begin{aligned}\tilde{z} &= E(z|z \geq z^*)P(z \geq z^*) + z^*P(z < z^*)\\ &= \int_{z^*}^{S} \frac{z}{S-R}dz + z^* \int_{R}^{z^*} \frac{1}{S-R}dz = \frac{1}{S-R}\left.\frac{z^2}{2}\right|_{z^*}^{S} + \frac{z^*}{S-R}\, z|_R^{z^*}\\ &= \frac{S^2 - z^{*2}}{2(S-R)} + \frac{z^*(z^*-R)}{S-R} = \frac{S^2 - 2Rz^* + z^{*2}}{2(S-R)}\end{aligned}$$

$$\frac{d\tilde{z}}{dz^*} = \frac{2z^* - 2R}{2(S-R)} > 0 \quad \forall z^* > R;$$

(c)

$$\int_{z^*}^{S} \frac{\exp(z)}{S-R}dz = \left.\frac{\exp(z)}{S-R}\right|_{z^*}^{S} = \frac{\exp(S) - \exp(z^*)}{S-R}.$$

Proof of Proposition 3.

Write expression (44) as

$$\frac{1-n_{1t}}{n_{1t}} = \frac{k_\pi}{k_X e^n}(1-h_{1t})h_{0t} \tag{A7}$$

and substitute this result into (35):

$$\begin{aligned}\tilde{V}(h_{0t}, X_{1t}^*) = \left[\frac{1}{1-\beta(1-k_X)}\right] \Bigg\{\log\left[\frac{(1-\tau)(1-\alpha_1)}{\alpha_1}\right]\\ - \log\left[\frac{k_\pi}{k_X e^n}\right] - \log(1-h_{1t}) - \log h_{0t} + \log \hat{X}_{1t} - \log \tilde{X}_{1t}\Bigg\}.\end{aligned} \tag{A8}$$

Condition (39) can be written

$$h_{Yt}(X_{1t}^*) = -k_U + k_\pi \beta \tilde{V}(h_{0t}, X_{1t}^*) \tag{A9}$$

where $h_{Yt}(X_{1t}^*)$ denotes the function of X_{1t}^* given in (38).

From rows (2) and (5) of Table 1, as $\log X_{1t}^*$ increases from R_t to S_t, the left side of (A9) monotonically increases from $-\infty$ to ∞. For fixed $h_{0t} > 0$, the right side monotonically decreases from ∞ to $-\infty$. Thus given any $h_{0t} \in (0,1)$, there exists a unique $\log X_{1t}^* \in (R_t, S_t)$ at which condition (A9) holds, that is, for which conditions (44) and (39) simultaneously hold. Denote this solution $X_{1t}^*(h_{0t})$.

From (A8), a larger value of h_{0t} lowers the right side of (A9) and thus is associated with a lower value of X_{1t}^*: $\partial X_{1t}^*(h_{0t})/\partial h_{0t} < 0$. As $h_{0t} \to 0$, $-\log h_{0t} \to \infty$ and $\log(X_{1t}^*(h_{0t}))$ is driven to S_t. Since $h_{Yt}(X_{1t}^*)$ in (38) is monotonically increasing in X_{1t}^* and since $X_{1t}^*(h_{0t})$ is monotonically decreasing in h_{0t}, it follows that $h_{Yt}(X_{1t}^*(h_{0t}))$ is a monotonically decreasing function of h_{0t}. By the definition of $X_{1t}^*(h_{0t})$, we know that

$$h_{Yt}(X_{1t}^*(h_{0t})) = -k_U + k_\pi \beta \tilde{V}(h_{0t}, X_{1t}^*(h_{0t})) \tag{A10}$$

holds for all h_{0t}. Monotonicity of the left side of (A10) as a function of h_{0t} implies that the right side is also a monotonically decreasing function of h_{0t}.

Next consider the incentives for applying for a position with continuing enterprises. Substituting (A7) into (41),

$$\pi_t(h_{0t}) = \max\left\{\frac{(1-k_X)(1-e^{-n})k_\pi}{k_X}\frac{h_{0t}}{(1-h_{0t})}, 1\right\}, \tag{A11}$$

allowing us to write (42) as

$$-k_U + \beta k_\pi \tilde{V}(h_{0t}, X_{1t}^*(h_{0t})) = \beta \pi_t(h_{0t}) \tilde{V}(h_{0t}, X_{1t}^*(h_{0t})). \tag{A12}$$

Recalling that $\tilde{V}(h_{0t}, X_{1t}^*(h_{0t}))$ is a monotonically decreasing function of h_{0t}, consider two cases. Suppose first that $\tilde{V}(h_{0t}, X_{1t}^*(h_{0t}))$ is positive at its lowest point ($h_{0t} = 1$). Note from (A11) that $\pi_t = 1$ at this point. With $\tilde{V}$ positive and $\pi_t = 1 > k_\pi$, the right side of (A12) must be larger than the left side at the lowest possible value for $\tilde{V}$, namely $\tilde{V}(1, X_{1t}^*(1))$. As h_{0t} decreases below 1, $\tilde{V}$ monotonically increases and π_t monotonically decreases, the latter eventually reaching 0 as $h_{0t} \to 0$. Thus there exists a unique value $h_{0t}^0 \in (0,1)$ at which (A12) holds; see the top panel of Figure A1. Call this value $\bar{h}_{0t}$. This value implies a unique $X_{1t}^*(\bar{h}_{0t})$, a unique $h_{1t}(X_{1t}^*(\bar{h}_{0t})$ and thus a unique $n_{1t}(\bar{h}_{0t})$ from (A7). By construction $(X_{1t}^*(h_{0t}^0, n_{1t}(h_{0t}^0), h_{0t}^0)$ satisfy (43), (42) and (44).

Alternatively, suppose that $\tilde{V}(1, X_{1t}^*(1))$ is negative (see the bottom panel of Figure A1). Since $\tilde{V}$ is monotonically decreasing in h_{0t} and goes to ∞ as $h_{0t} \to 0$, there exists a unique $\bar{h}_{0t} \in (0,1)$ at which $\tilde{V}(\bar{h}_{0t}, X_{1t}^*(\bar{h}_{0t})) = 0$. At this point the right side of (A12) is zero and the left side is negative. As h_{0t} decreases below $\bar{h}_{0t}$, the value of $\tilde{V}$ increases without bound while the magnitude $\pi_t(h_{0t})$ eventually goes to 0. Thus there again exists a unique h_{0t}^0 for which condition (A12) holds and for which $(X_{1t}^*(h_{0t}^0, n_{1t}(h_{0t}^0), h_{0t}^0)$ simultaneously satisfies (44), (42) and (39).

Proof of Proposition 4.

$$h_{1,t+1} = \frac{S_{t+1} - \log X_{1,t+1}^*}{S_{t+1} - R_{t+1}} = \frac{S_t + g - (\log X_{1t}^* + g)}{(S_t + g) - (R_t + g)} = \frac{S_t - \log X_{1t}^*}{S_t - R_t} = h_{1t}$$

$$\begin{aligned}
\log \tilde{X}_{1,t+1} &= \frac{(S_t + g)^2 - 2(R_t + g)(\log X_{1t}^* + g) + (\log X_{1t}^* + g)^2}{2[(S_t + g) - (R_t + g)]} \\
&= \frac{S_t^2 - 2R_t \log X_{1t}^* + (\log X_{1t}^*)^2}{2(S_t - R_t)} + \frac{2S_t g + g^2 - 2g \log X_{1t}^* - 2gR_t - 2g^2 + 2g \log X_{1t}^* + g^2}{2(S_t - R_t)} \\
&= \log \tilde{X}_{1t} + \frac{g(2S_t - 2R_t)}{2(S_t - R_t)} = \log \tilde{X}_{1t} + g
\end{aligned}$$

$$\hat{X}_{1,t+1} = \frac{\exp(S_t + g) - X_{1t}^* \exp(g)}{(S_t + g) - (R_t + g)} = \exp(g) \left[\frac{\exp(S_t) - X_{1t}^*}{S_t - R_t} \right] = \exp(g) \hat{X}_{1t}.$$

The other results follow immediately.

Proof of Proposition 5.

(a) Let $(X_{1t_0}^*, n_1^0, h_0^0)$ be the unique solution to (39), (42) and (44) for date t_0. Then $(e^g X_{1t_0}^*, n_1^0, h_0^0)$ solve these three equations for date $t_0 + 1$, as can be verified as follows. From Proposition 4, $X_{1,t_0+1}^* = e^g X_{1t_0}^*$ would imply $h_{1,t_0+1} = h_1^0$, $\log \hat{X}_{1,t_0+1} = g + \log \hat{X}_{1t_0}$ and $\log \tilde{X}_{1,t_0+1} = g + \log \tilde{X}_{1t_0}$ establishing from (38) that $h_{Y,t_0+1} = h_{Yt_0}$ and from (35) that $\tilde{V}_{t_0+1} = \tilde{V}_{t_0}$. Hence (39), (42) and (44) are all satisfied at date $t_0 + 1$, confirming that $(e^g X_{1t_0}^*, n_1^0, h_0^0)$ is the solution. By induction, $(e^{g(t-t_0)} X_{1t_0}^*, n_1^0, h_0^0)$ is the solution for all t.

(b) Of the $J_{2t_0} = k_J/k_X$ goods at initial date t_0, $k_X J_{2t_0} = k_J$ will no longer be produced beginning in $t_0 + 1$. And since $h_{0t_0} > 0$, k_J new goods (one of each type) will begin being produced in $t_0 + 1$. Thus $J_{2,t_0+1} = J_{2t_0}$ and by induction J_{2t} is constant for all t.

(c) Along the steady-state growth path, a fraction $(1-\alpha_1)(1-\tau)$ of total income $\bar{Y}_t$ is earned by specialized workers and the remaining $[\alpha_1 + \tau(1-\alpha_1)]\bar{Y}_t$ is received by nonspecialized. Each of these groups on average spends a fraction α_{jt} of their income on good j. Since $n_{jt} N_t X_{jt}$ units of good j get produced, $(1 - \alpha_1)(1 - \tau) n_{jt} N_t X_{jt}$ units of good j are consumed by the specialized and the remaining $[\alpha_1 + \tau(1 - \alpha_1)] n_{jt} N_t X_{jt}$ by nonspecialized. Dividing the first expression by the total number of specialized workers $(1 - n_{1t}) N_t$ gives result (c). Result (h) below verifies that this is in fact the same number for all specialized workers.

(d) Dividing nonspecialized total spending on j, $[\alpha_1 + \tau(1 - \alpha_1)] n_{jt} N_t X_{jt}$, by the total number of nonspecialized $n_1^0 N_t$ gives result (d). Since productivities x_{it} are drawn independently over time, this is the average nonspecialized spending and is the level of consumption q_{njt}^0 along the steady-state growth path.

(e) With $n_{jt} = n_j^0$, consumption of good j per individual in (50)-(51) grows at rate g so that total consumption Q_{jt} grows at $g + n$ and equals total production in (32).

(f) The ratio of nominal spending on good j to that for good 1 is $(P_{jt} Q_{jt})/(P_{1t} Q_{1t}) = \alpha_j/\alpha_1$. Since $Q_{jt} = n_j^0 N_t X_{jt}$, $P_{jt}/P_{1t} = (\alpha_j n_{1t} \hat{X}_{1t})/(\alpha_1 n_j^0 X_{jt})$. Since $\hat{X}_{1t}$ and X_{jt} both grow at rate g, the ratio $\hat{X}_{1t}/X_{jt}$ is constant over time.

(g) This follows from applying results (c) and (d) to expression (9).

(h) Total spending on good j is $P_{jt} Q_{jt} = \alpha_j Y_t$, so the after-tax income per person producing good j is

$$\frac{(1 - \tau) P_{jt} Q_{jt}}{n_{jt} N_t} = \frac{\alpha_j Y_t (1 - \tau)}{n_{jt} N_t}. \tag{A13}$$

Equation (49) establishes that at date t_0 this magnitude is

$$\frac{(1 - \tau) P_{jt_0} Q_{jt_0}}{n_{jt_0} N_{t_0}} = \frac{Y_{t_0} (1 - \tau)(1 - \alpha_1)}{(1 - n_1^0) N_{t_0}}$$

which is the same for all $j \in \mathcal{J}_{2t_0}$. Thus the stated initial conditions imply that all specialized workers earn the same income at date t_0. The income for a worker producing good j at date t is $P_{jt}X_{jt}$, which from result (f) is $e^{g(t-t_0)}$ times the income that individual received at date t_0, the same constant factor for each j.

For goods that are produced for the first time in period t, substituting condition (45) into (A13) gives

$$\frac{(1-\tau)P_{jt}Q_{jt}}{n_{jt}N_t} = \frac{(1-\alpha_1)Y_t(1-\tau)}{(1-n_1^0)N_t} \quad j \in \mathcal{J}_{2t}^{\sharp},$$

which is the same for each $j \in \mathcal{J}_{2t}^{\sharp}$ and the same as the income received by those producing continuing specialized goods.

Proof of Proposition 6.

(a) Note that

$$\bar{q}_{ijt} = \begin{cases} 2\chi_{jt}q_{sjt}^0 & \text{for } i \in \mathcal{M}_{st} \\ 2\chi_{jt}q_{njt}^0 & \text{for } i \in \mathcal{M}_{nt} \end{cases} \tag{A14}$$

where $\mathcal{M}_{st}$ and $\mathcal{M}_{nt}$ denote the sets of specialized and nonspecialized workers, respectively. From (A14), (9), and Proposition 5c-e:

$$\bar{Q}_{jt} = 2\chi_{jt}[n_{1t}q_{njt}^0 + (1-n_{1t})q_{sjt}^0]N_t = 2\chi_{jt}H_t n_j^0 X_{jt}^0 N_t \tag{A15}$$

$$H_t = \frac{n_{1t}[\alpha_1 + \tau(1-\alpha_1)}{n_1^0} + \frac{(1-n_{1t})(1-\alpha_1)(1-\tau)}{1-n_1^0} = 1 + \lambda_H(n_{1t} - n_1^0).$$

(b) This simply restates (24) and (10).

(c) For $\gamma_{ijt} = \xi_{jt}\alpha_{jt}/(q_{ijt}^0)^2$ and $\bar{q}_{ijt} = 2\chi_{jt}q_{ijt}^0$ consumer i's first-order condition (3) is

$$\frac{\xi_{jt}\alpha_{jt}}{(q_{ijt}^0)^2}(2\chi_{jt}q_{ijt}^0 - q_{ijt}) = \lambda_{it}P_{jt}. \tag{A16}$$

From (50)-(52),

$$\frac{q_{ijt}^0}{q_{i1t}^0} = \frac{n_j^0 X_{jt}^0}{n_1^0 \hat{X}_{1t}^0} = \frac{\alpha_j^0 P_1^0}{\alpha_1 P_j^0},$$

allowing (A16) to be written

$$\xi_{jt}\alpha_{jt}(2\chi_{jt}q_{ijt}^0 - q_{ijt}) = \lambda_{it}P_{jt}\left(\frac{\alpha_j^0 P_1^0}{\alpha_1 P_j^0}\right)^2 (q_{i1t}^0)^2. \tag{A17}$$

From (A14), $\int_0^{N_t} 2\chi_{jt}q_{ijt}^0 di = \bar{Q}_{jt}$. Thus integrating (A17) over i gives

$$\xi_{jt}\alpha_{jt}(\bar{Q}_{jt} - Q_{jt}) = \Lambda_t P_{jt}\left(\frac{\alpha_j^0 P_1^0}{\alpha_1 P_j^0}\right)^2 \tag{A18}$$

for $\Lambda_t = \int_0^{N_t} \lambda_{it}(q^0_{i1t})^2 di$. Dividing (A18) by its value for $j = 1$,

$$\frac{\xi_{jt}\alpha_{jt}(\bar{Q}_{jt} - Q_{jt})}{\alpha_1(\bar{Q}_{1t} - Q_{1t})} = \left(\frac{\alpha^0_j P^0_1}{\alpha_1 P^0_j}\right)^2 \left(\frac{P_{jt}}{P_{1t}}\right).$$

Rearranging gives (58).

If we substitute (60), $Q_{jt} = \bar{Q}_{jt}/2$, and (54) into (58) we get

$$\begin{aligned}
\frac{P_{jt}}{P_{1t}} &= \xi_{jt}\left(\frac{P^0_j}{P^0_1}\right)^2 \left(\frac{\alpha_1}{\alpha^0_j}\right)\left(\frac{n_{jt}(1-n^0_1)}{n^0_j(1-n_{1t})}\right)\left[\frac{\chi_{jt}H_t n^0_j X^0_{jt} N_t}{2H_t n^0_1 \hat{X}^0_{1t} N_t - n_{1t}\hat{X}_{1t}N_t}\right]\\
&= \xi_{jt}\left(\frac{P^0_j}{P^0_1}\right)^2 \left(\frac{\alpha_1}{\alpha^0_j}\right)\left(\frac{n^0_j X^0_{jt}}{n^0_1 \hat{X}^0_{1t}}\right)\left(\frac{n_{jt}(1-n^0_1)}{n^0_j(1-n_{1t})}\right)\left[\frac{\chi_{jt}H_t n^0_1 \hat{X}^0_{1t}}{2H_t n^0_1 \hat{X}^0_{1t} - n_{1t}\hat{X}_{1t}}\right]\\
&= \xi_{jt}\left(\frac{P^0_j}{P^0_1}\right)\left(\frac{n_{jt}(1-n^0_1)}{n^0_j(1-n_{1t})}\right)\left[\frac{\chi_{jt}H_t n^0_1 \hat{X}^0_{1t}}{2H_t n^0_1 \hat{X}^0_{1t} - n_{1t}\hat{X}_{1t}}\right].
\end{aligned}$$

The last equality followed from (52) and establishes (59).

(d) This is obtained by taking the ratio of (53) to (45).

(e) Expression (62) follows from (54):

$$\frac{\bar{Q}_{j,t+1}/2}{X_{j,t+1}} = \frac{\chi_{j,t+1}H_{t+1}n^0_j X^0_{j,t+1}N_{t+1}}{X_{j,t+1}} = \chi_{j,t+1}H_{t+1}\left(\frac{X^0_{j,t+1}}{X_{j,t+1}}\right)N^0_{j,t+1}.$$

(f) From (58) and (60),

$$Y_{jt} = (1-\tau)P_{1t}\left(\frac{P^0_j}{P^0_1}\right)^2\left(\frac{\alpha_1}{\alpha^0_j}\right)\left(\frac{n_{jt}(1-n^0_1)}{n^0_j(1-n_{1t})}\right)\xi_{jt}\left[\frac{\bar{Q}_{jt}-Q_{jt}}{\bar{Q}_{1t}-Q_{1t}}\right]\frac{Q_{jt}}{n_{jt}N_t}. \tag{A19}$$

From (52) we know

$$\frac{P^0_j}{P^0_1} = \frac{\alpha^0_j n^0_1 \hat{X}^0_{1t}}{\alpha_1 n^0_j X^0_{jt}} = \frac{\alpha^0_j Q^0_{1t}}{\alpha_1 Q^0_{jt}} \tag{A20}$$

allowing (A19) to be written

$$\frac{Y_{jt}}{P_{1t}} = (1-\tau)\frac{Q^0_{1t}}{Q^0_{jt}}\frac{P^0_j}{P^0_1}\frac{1-n^0_1}{1-n_{1t}}\xi_{jt}\left[\frac{\bar{Q}_{jt}-Q_{jt}}{\bar{Q}_{1t}-Q_{1t}}\right]\frac{Q_{jt}}{n^0_j N_t}. \tag{A21}$$

Using (45) we can also conclude from (A20) that $P^0_j/P^0_1 = [(1-\alpha_1)n^0_1\hat{X}^0_{1t}]/[\alpha_1(1-n^0_1)X^0_{jt}]$. Substituting this into (A21) and rearranging,

$$\frac{Y_{jt}}{P_{1t}} = (1-\tau)\frac{(1-\alpha_1)n^0_1}{\alpha_1(1-n^0_1)}\frac{1-n^0_1}{1-n_{1t}}\xi_{jt}\frac{(\bar{Q}_{jt}-Q_{jt})/Q^0_{jt}}{(\bar{Q}_{1t}-Q_{1t})/Q^0_{1t}}\frac{Q_{jt}}{n^0_j N_t X^0_{jt}}\hat{X}^0_{1t}. \tag{A22}$$

Note that if $Q_{jt} = \bar{Q}_{jt}/2$,

$$Q_{jt}(\bar{Q}_{jt} - Q_{jt}) = (\bar{Q}_{jt}/2)^2 = (\chi_{jt} H_t n_j^0 X_{jt}^0 N_t)^2 = (\chi_{jt} H_t Q_{jt}^0)^2. \tag{A23}$$

Also

$$\bar{Q}_{1t} - Q_{1t}^0 = 2H_t Q_{1t}^0 - n_{1t}\hat{X}_{1t}N_t. \tag{A24}$$

Substituting (A23) and (A24) into (A22) results in

$$\frac{Y_{jt}}{P_{1t}} = (1-\tau)\frac{(1-\alpha_1)n_1^0}{\alpha_1(1-n_1^0)}\frac{1-n_1^0}{1-n_{1t}}\xi_{jt}\frac{(\chi_{jt}H_t)^2 Q_{1t}^0}{(2H_t Q_{1t}^0 - n_{1t}\hat{X}_{1t}N_t)}\hat{X}_{1t}^0. \tag{A25}$$

Along the steady-state growth path, $n_{1t} = n_1^0$, and $\xi_{jt} = \chi_{jt} = H_t = 1$. Thus from (A25) the steady-state real income of specialized workers is given by (65). Substituting (65) into (A22) and (A25) gives (63) and (64).

Results (g)-(k) restate expressions from elsewhere in the paper.

(l) Notice from $\alpha_j^0 = P_{jt}^0 Q_{jt}^0 / \sum_{j\in\mathcal{J}_t} P_{jt}^0 Q_{jt}^0$ that

$$Q_t = \frac{\sum_{j\in\mathcal{J}_t} P_{jt}^0 Q_{jt}^0 (Q_{jt}/Q_{jt}^0)}{\sum_{j\in\mathcal{J}_t} P_{jt}^0 Q_{jt}^0} = \sum\nolimits_{j\in\mathcal{J}_t} \alpha_j^0 (Q_{jt}/Q_{jt}^0).$$

Recall also that for $j \in \mathcal{J}_{2t}$, $Q_{jt}^0 = n_j^0 N_t X_{jt}^0$. Using this along with (45) and (24) we conclude that

$$\begin{aligned}\sum\nolimits_{j\in\mathcal{J}_t} \alpha_j^0 (Q_{jt}/Q_{jt}^0) &= \sum\nolimits_{j\in\mathcal{J}_{2t}} \left(\frac{\alpha_j^0}{n_j^0}\right)\left(\frac{Q_{jt}}{N_t X_{jt}^0}\right) + \alpha_1\left(\frac{Q_{1t}}{Q_{1t}^0}\right) \\ &= \left(\frac{1-\alpha_1}{1-n_1^0}\right)\sum\nolimits_{j\in\mathcal{J}_{2t}}\left(\frac{Q_{jt}}{N_t X_{jt}^0}\right) + \left(\frac{\alpha_1}{n_1^0}\right)\left(\frac{\hat{X}_{1t}}{\hat{X}_{1t}^0}\right)n_{1t}.\end{aligned}$$

Note that if $\chi_{jt} = \zeta_{jt} = 1$ and $Q_{jt} = \bar{Q}_{jt}/2$, then $Q_{jt} = H_t n_j^0 N_t X_{jt}^0$ so $Q_{jt}/(N_t X_{jt}^0) = H_t n_j^0$ and (73) becomes (74).

Appendix B. Linearized adjustment dynamics (online)

Define $w_t^\dagger = \log w_t - \log w_t^0$ for $w_t = \bar{Q}_{jt}, Q_{jt}, X_{1t}^*, Y_{jt}, C_t, X_{jt}, Q_t, \chi_{jt}, \xi_{jt}, \zeta_{jt}$ (recalling that $\log Q_t^0 = \log \chi_{jt}^0 = \log \xi_{jt}^0 = \log \zeta_{jt}^0 = 0$); $w_t^\dagger = w_t - w^0$ for $w_t = \alpha_{jt}, n_{jt}, h_{0t}, \tilde{V}_{jt}$; $P_{jt}^\dagger = \log(P_{jt}/P_{1t}) - \log(P_{jt}^0/P_{1t}^0)$; $\tilde{Y}_{jt}^\dagger = \log Y_{jt} - \log Y_{jt}^0 - [\log(P_{1t}\tilde{X}_{1t}) - \log(P_{1t}^0 \tilde{X}_{1t}^0)]$; $\lambda_2, \lambda_3, \lambda_5$ are the derivatives in Table 1 and λ_H the derivative in Proposition 6a.

Linearized version of Proposition 6.

Evaluating derivatives of Proposition 6 along the steady-state growth path and taking deviations from steady state results in

$$\bar{Q}_{jt}^\dagger = \chi_{jt}^\dagger + \lambda_H n_{1t}^\dagger \quad j \in \mathcal{J}_t \tag{B1}$$

$$Q_{1t}^\dagger = \frac{1}{n_1^0} n_{1t}^\dagger + \lambda_5 X_{1t}^{*\dagger} \tag{B2}$$

$$Q_{jt}^\dagger = \begin{cases} \chi_{jt}^\dagger + \lambda_H n_{1t}^\dagger & \text{if } j \in \mathcal{J}_{2t} \text{ and } \bar{Q}_{jt}/2 \le n_{jt} N_t X_{jt} \\ \frac{n_{jt}^\dagger}{n_j^0} + \zeta_{jt}^\dagger & \text{if } j \in \mathcal{J}_{2t} \text{ and } \bar{Q}_{jt}/2 > n_{jt} N_t X_{jt} \end{cases} \tag{B3}$$

$$p_{jt}^\dagger = \frac{\alpha_{jt}^\dagger}{\alpha_j^0} + \xi_{jt}^\dagger + 2\chi_{jt}^\dagger - Q_{jt}^\dagger + \frac{n_{1t}^\dagger}{n_1^0} + \lambda_5 X_{1t}^{*\dagger} \quad j \in \mathcal{J}_{2t} \tag{B4}$$

$$\frac{\alpha_{jt}^\dagger}{\alpha_j^0} = \frac{n_{jt}^\dagger}{n_j^0} + \frac{1}{1 - n_1^0} n_{1t}^\dagger \quad j \in \mathcal{J}_{2t} \tag{B5}$$

$$n_{j,t+1}^\dagger = \begin{cases} n_j^0 \chi_{j,t+1}^\dagger + n_j^0 \lambda_H n_{1,t+1}^\dagger - n_j^0 \zeta_{j,t+1}^\dagger & \text{if } j \in \mathcal{J}_{2t}^\natural \text{ and } \bar{Q}_{j,t+1}/2 \ge X_{j,t+1} N_{jt} \\ n_{jt}^\dagger - n & \text{if } j \in \mathcal{J}_{2t}^\natural \text{ and } \bar{Q}_{j,t+1}/2 < X_{j,t+1} N_{jt} \end{cases} \tag{B6}$$

$$Y_{jt}^\dagger = \xi_{jt}^\dagger + \frac{1}{n_1^0(1 - n_1^0)} n_{1t}^\dagger + 2\chi_{jt}^\dagger + \lambda_5 X_{1t}^{*\dagger} \quad j \in \mathcal{J}_{2t} \tag{B7}$$

$$\tilde{V}_{jt}^\dagger = Y_{jt}^\dagger - \lambda_3 X_{1t}^{*\dagger} + \beta(1 - k_X)\tilde{V}_{j,t+1}^\dagger \quad j \in \mathcal{J}_{2t}. \tag{B8}$$

To derive result (B4) we used the fact that for all $j \in \mathcal{J}_t$, $\bar{Q}_{jt}^0 = 2Q_{jt}^0$ establishing

$$\begin{aligned} \log(\bar{Q}_{jt} - Q_{jt}) &\simeq \log Q_{jt}^0 + \frac{1}{Q_{jt}^0}[(\bar{Q}_{jt} - \bar{Q}_{jt}^0) - (Q_{jt} - Q_{jt}^0)] \\ &= \log Q_{jt}^0 + \frac{2}{\bar{Q}_{jt}^0}(\bar{Q}_{jt} - \bar{Q}_{jt}^0) - \frac{Q_{jt} - Q_{jt}^0}{Q_{jt}^0} \\ &= \log Q_{jt}^0 + 2\bar{Q}_{jt}^\dagger - Q_{jt}^\dagger \end{aligned} \tag{B9}$$

$$p_{jt}^\dagger = \frac{\alpha_{jt}^\dagger}{\alpha_j^0} + \xi_{jt}^\dagger + 2\bar{Q}_{jt}^\dagger - Q_{jt}^\dagger - 2\bar{Q}_{1t}^\dagger + Q_{1t}^\dagger.$$

Result (B4) then follows from (B1) and (B2). Similarly to derive (B7) we used (B9) along with

$$Y_{jt}^{\dagger} = \xi_{jt}^{\dagger} + \frac{1}{1-n_1^0} n_{1t}^{\dagger} + Q_{jt}^{\dagger} + 2\bar{Q}_{jt}^{\dagger} - Q_{jt}^{\dagger} - 2\bar{Q}_{1t}^{\dagger} + Q_{1t}^{\dagger}.$$

For $\tilde{Y}_{jt} = \log Y_{jt} - \log(P_{1t}\tilde{X}_{1t})$ and $\tilde{Y}_{jt}^{\dagger} = Y_{jt}^{\dagger} - \tilde{X}_{1t}^{\dagger} = Y_{jt}^{\dagger} - \lambda_3 X_{1t}^{*\dagger}$ it follows from (B7) that

$$\tilde{Y}_{jt}^{\dagger} = \xi_{jt}^{\dagger} + 2\chi_{jt}^{\dagger} + \frac{1}{n_1^0(1-n_1^0)} n_{1t}^{\dagger} + (\lambda_5 - \lambda_3) X_{1t}^{*\dagger} \quad j \in \mathcal{J}_{2t}. \tag{B10}$$

Appendix C. Details of dynamic solution (online)

Baseline model and Examples 9.1-9.4.

Here we write the system in the form of 10 dynamic equations (C1)-(C10) in the 10 endogenous variables

$$z_t = (n_{1t}, \tilde{V}_t, \bar{n}_t, y_t, c_t, x^*_{1t}, \pi_t, h_{0t}, n^\sharp_{t+1}, n^\natural_{t+1})'$$

with four exogenous shocks $s_{1t}, s_{3t}, s_{6t}, s_{7t}$. In addition we will use the following symbols to simplify some of the expressions, to be substituted in when coded:

$$H_t = 1 + \lambda_H(n_{1t} - n^0_1)$$

$$h_{1t} = \frac{S - x^*_{1t}}{S - R}.$$

The system can be written as follows:

$$n_{1,t+1} = 1 - n^\sharp_{t+1} - n^\natural_{t+1} \tag{C1}$$

$$\tilde{V}_t = \log y_t - \left[\frac{S^2 - 2Rx^*_{1t} + (x^*_{1t})^2}{2(S-R)}\right] + \beta(1-k_X)\tilde{V}_{t+1} \tag{C2}$$

$$\bar{n}_{t+1} = (1-k_X)\bar{n}_t s_{1t} + n^\sharp_{t+1}. \tag{C3}$$

In the baseline model and Examples 9.1-9.2, $s_{1t} = 1$ for all t. For Examples 9.3-9.4, since a fraction κ of the specialists drop out after t_0, the value of $\bar{n}_{t_0+1}$ is characterized by

$$\bar{n}_{t_0+1} = (1-k_X)(1-\kappa)\bar{n}_{t_0} + n^\sharp_{t_0+1}$$

which is implemented by setting

$$s_{1t} = \begin{cases} 1-\kappa & t = t_0 \\ 1 & t = t_0+1, t_0+2, \ldots \end{cases}.$$

$$y_t = \frac{y^0(1-n^0_1)H^2_t q^0_1}{(1-n_{1t})\left\{2H_t q^0_1 - n_{1t}\left[\frac{\exp(S)-\exp(x^*_{1t})}{S-R}\right]\right\}} \tag{C4}$$

$$c_t = \frac{\tau(1-n_{1t})y_t}{n_{1t}(1-h_{1t})(1-\tau)} s_{3t}. \tag{C5}$$

For the baseline model, $s_{3t} = 1$ for all t. In Examples 9.1-9.4, a fraction κ of the goods are impacted at t_0 and none are impacted afterwards, so

$$s_{3t} = \begin{cases} 1 + \kappa(\chi^2 - 1) & t = t_0 \\ 1 & t = t_0 + 1, t_0 + 2, ... \end{cases}.$$

$$x_{1t}^* - \log c_t = -k_U + \beta k_\pi \tilde{V}_{t+1} \tag{C6}$$

$$x_{1t}^* - \log c_t = \beta \pi_t \tilde{V}_{t+1} \tag{C7}$$

$$\pi_t = \frac{n_{t+1}^\natural e^n - (1 - k_X)(1 - n_{1t}) s_{6t}}{(1 - h_{1t})(1 - h_{0t}) n_{1t}}. \tag{C8}$$

For the baseline model and Examples 9.1-9.2, $s_{6t} = 1$. For Examples 9.3-9.4, a fraction κ discontinue after period t_0, as represented by

$$s_{6t} = \begin{cases} (1 - \kappa) & t = t_0 \\ 1 & t = t_0 + 1, t_0 + 2, ... \end{cases}.$$

$$n_{t+1}^\sharp = e^{-n}(1 - h_{1t}) h_{0t} n_{1t} k_\pi \tag{C9}$$

$$n_{t+1}^\natural = H_{t+1}(1 - k_X) s_{7t} \bar{n}_t. \tag{C10}$$

For the baseline model and Examples 9.1-9.2, $s_{7t} = 1$ for all t. For Examples 9.3-9.4,

$$s_{7t} = \begin{cases} 1 - \kappa & t = t_0 \\ 1 & t = t_0 + 1, t_0 + 2, ... \end{cases}.$$

Predetermined variables at date t_0 are $n_{1t_0} = n_1^0$, $\bar{n}_{t_0} = 1 - n_1^0$, $n_{t_0}^\sharp$, and $n_{t_0}^\natural$. Initial values of $n_{t_0}^\sharp$ and $n_{t_0}^\natural$ do not appear anywhere in the system. A solution is a sequence $\{z_t\}_{t=t_0}^T$ for very large T satisfying $n_{1t_0} = n_1^0$, $\bar{n}_{t_0} = 1 - n_1^0$ and $n_{1T} \simeq n_1^0, \tilde{V}_T \simeq \tilde{V}^0, \bar{n}_T \simeq 1 - n_1^0, y_T \simeq y^0, c_T \simeq c^0, x_{1T}^* \simeq x_1^{*0}, \pi_T \simeq \pi^0, h_{0T} \simeq h_0^0, n_{T+1}^\sharp \simeq n^{\sharp 0}, n_{T+1}^\natural \simeq n^{\natural 0}$.

Example 9.5.

In this example we need to keep track of the fraction of the population specializing in impacted and nonimpacted goods (n_t^χ and n_t^c, respectively) and what the fractions would be if each good employed its steady-state level n_j^0 ($\bar{n}_t^\chi = \sum_{j \in \mathcal{J}_{2t}^\chi} n_j^0$ and $\bar{n}_t^c = \sum_{j \in \mathcal{J}_{2t}^c} n_j^0$). The value of $\bar{n}_t^c$ evolves independently of all other variables, since goods in J_{2t}^c started out with $n_{jt_0} = n_j^0$ and a fraction k_X of these disappear each period,

$$\bar{n}_{t+1}^c = (1 - k_X)\bar{n}_t^c \quad t = t_0, ..., t_0 + D - 2$$

starting from $\bar{n}_{t_0}^c = (1 - \kappa)(1 - n_1^0)$. The other three magnitudes $(n_t^\chi, n_t^c, \bar{n}_t^\chi)$ influence and respond to other variables during the initial periods as described below. We can adapt the structure used for Examples 9.1-9.4 to this case by reinterpreting the meaning of $n_t^\natural$ and $\bar{n}_t$ over the initial periods and by adding an eleventh state variable to the system, n_t^{χ}, which denotes the fraction of the population producing demand-impacted goods at t. Notice $n_{t_0}^{\chi} = \kappa(1 - n_1^0)$ and $n_{t_0+D}^{\chi} = 0$.

During the impacted period, the variable $n_t^{\natural}$ will represent the fraction of workers producing nonimpacted goods. Thus specialized workers during the impacted phase consist of nonimpacted workers $n_t^{\natural}$ plus impacted workers n_t^{χ}. After the impacted period, $n_t^{\natural}$ will revert to its original interpretation as number of continuing workers. Thus

$$n_{1,t+1} = \begin{cases} 1 - n_{t+1}^{\natural} - n_{t+1}^{\chi} & \text{for } t = t_0, t_0+1, ..., t_0+D-2 \\ 1 - n_{t+1}^{\natural} - n_{t+1}^{\sharp} & \text{for } t = t_0+D-1, t_0+D, ... \end{cases} .$$

Note that when $t = t_0 + D - 1$, it will be the case that $n_{1,t+1} = n_{1,t_0+D}$ for which there are no impacted workers. This can be written in terms of shocks as

$$n_{1,t+1} = 1 - n_{t+1}^{\natural} - s_{8t} n_{t+1}^{\chi} - (1 - s_{8t}) n_{t+1}^{\sharp} \tag{C11}$$

$$s_{8t} = \begin{cases} 1 & \text{for } t = t_0, t_0+1, ..., t_0+D-2 \\ 0 & \text{for } t = t_0+D-1, t_0+D, ... \end{cases} .$$

Equations (C2)-(C4) for Example 9.5 are the same as in Examples 9.1-9.2:

$$\tilde{V}_t = \log y_t - \left[\frac{S^2 - 2Rx_{1t}^* + (x_{1t}^*)^2}{2(S-R)} \right] + \beta(1 - k_X)\tilde{V}_{t+1} \tag{C12}$$

$$\bar{n}_{t+1} = (1 - k_X)\bar{n}_t + n_{t+1}^{\sharp} \tag{C13}$$

$$y_t = \frac{y^0(1 - n_1^0) H_t^2 q_1^0}{(1 - n_{1t}) \left\{ 2H_t q_1^0 - n_{1t} \left[\frac{\exp(S) - \exp(x_{1t}^*)}{S-R} \right] \right\}} \tag{C14}$$

though (C13) will not be referenced by the other equations during the impacted period. If n_t^c denotes the number of nonimpacted workers, the general expression for unemployment compensation is

$$\begin{aligned} c_t &= \frac{\tau y_t (n_t^c + n_t^{\chi} \chi^2)}{n_{1t}(1 - h_{1t})(1 - \tau)} \\ &= \begin{cases} \frac{c^0[(1-\kappa) + \kappa\chi^2](1 - h_1^0)}{1 - h_{1t}} & \text{for } t = t_0 \\ \frac{\tau y_t (n_t^{\natural} + n_t^{\chi}\chi^2)}{n_{1t}(1-h_{1t})(1-\tau)} & \text{for } t = t_0+1, t_0+2, ..., t_0+D-1 \\ \frac{\tau y_t (1 - n_{1t})}{n_{1t}(1-h_{1t})(1-\tau)} & \text{for } t = t_0+D, t_0+D+1, ... \end{cases} . \end{aligned}$$

This can be written in terms of shocks as

$$c_t = s_{9t} \frac{c^0[(1-\kappa) + \kappa\chi^2](1 - h_1^0)}{1 - h_{1t}} + (1 - s_{9t}) \frac{\tau y_t [s_{10,t}(n_t^{\natural} + n_t^{\chi}\chi^2) + (1 - s_{10,t})(1 - n_{1t})]}{n_{1t}(1 - h_{1t})(1 - \tau)} \tag{C15}$$

$$s_{9t} = \begin{cases} 1 & \text{for } t = t_0 \\ 0 & \text{for } t = t_0+1, t_0+2, ... \end{cases}$$

$$s_{10,t} = \begin{cases} 1 & \text{for } t = t_0, t_0+1, ..., t_0+D-1 \\ 0 & \text{for } t = t_0+D, t_0+D+1, ... \end{cases} .$$

The demand shock affects all newly created goods during the impacted period, so (C6) becomes

$$x_{1t}^* - \log c_t = -k_U + \beta k_\pi \tilde{V}_{t+1} + s_{5t} \tag{C16}$$

$$s_{5t} = \begin{cases} \beta k_\pi \sum_{s=1}^{t_0-D-t-1} [\beta(1-k_X)]^s \log \chi^2 & t = t_0, t_0+1, ..., t_0+D-2 \\ 0 & t = t_0+D-1, t_0+D, ... \end{cases} .$$

Expression (C7) continues as before

$$x_{1t}^* - \log c_t = \beta \pi_t \tilde{V}_{t+1} \tag{C17}$$

where π_t is now characterized by

$$\pi_t = \begin{cases} \frac{n_{t+1}^\natural e^n - (1-k_X)(1-\kappa)(1-n_1^0)}{(1-h_{1t})(1-h_{0t})n_{1t}} & t = t_0 \\ \frac{n_{t+1}^\natural e^n - (1-k_X)n_t^\natural}{(1-h_{1t})(1-h_{0t})n_{1t}} & t = t_0+1, t_0+2, ..., t_0+D-2 \\ \frac{n_{t+1}^\natural e^n - (1-k_X)(1-n_{1t})}{(1-h_{1t})(1-h_{0t})n_{1t}} & t = t_0+D-1, t_0+D, ... \end{cases} .$$

In the shock notation,

$$\pi_t = \frac{n_{t+1}^\natural e^n - (1-k_X)[s_{9t}(1-\kappa)(1-n_1^0) + (1-s_{9t})s_{8t}n_t^\natural + (1-s_{9t})(1-s_{8t})(1-n_{1t})]}{(1-h_{1t})(1-h_{0t})n_{1t}}. \tag{C18}$$

Expression (C9) remains unchanged:

$$n_{t+1}^\sharp = e^{-n}(1-h_{1t})h_{0t}n_{1t}k_\pi. \tag{C19}$$

The hiring decisions of continuing goods are characterized by

$$n_{t+1}^\natural = \begin{cases} H_{t+1}(1-k_X)\bar{n}_t^c & \text{for } t = t_0, t_0+1, ..., t_0+D-2 \\ H_{t+1}(1-k_X)\bar{n}_t & \text{for } t = t_0+D-1, t_0+D, ... \end{cases}$$

with $\bar{n}_t^c = (1-k_X)^{t-t_0}(1-\kappa)(1-n_1^0)$, or

$$n_{t+1}^\natural = H_{t+1}(1-k_X)[s_{8t}s_{11,t} + (1-s_{8t})\bar{n}_t] \tag{C20}$$

for $s_{11,t} = \bar{n}_t^c$. Since impacted goods do no hiring and new goods enter as impacted, the number of impacted workers over $t = t_0, t_0 + 1, ..., t_0 + D - 2$ evolves according to

$$n^{\chi}_{t+1} = e^{-n}(1 - k_X)n^{\chi}_t + n^{\sharp}_{t+1}$$

starting from $n^{\chi}_{t_0} = \kappa(1 - n^0_1)$. Thus

$$n^{\chi}_{t+1} = s_{8t}[n^{\sharp}_{t+1} + s_{9t}e^{-n}(1 - k_X)\kappa(1 - n^0_1) + (1 - s_{9t})e^{-n}(1 - k_X)n^{\chi}_t]. \tag{C21}$$

Real GDP.

Real GDP for Examples 9.1-9.5. If $\mathcal{J}^{\chi}_{2t}$ denote the set of demand-impacted goods and $\mathcal{J}^{c}_{2t}$ non-impacted specialized goods.

$$\frac{Q_{jt}}{N_t X^0_{jt}} = \begin{cases} \chi H_t n^0_j & \text{for } j \in \mathcal{J}^{\chi}_{2t} \\ H_t n^0_j & \text{for } j \in \mathcal{J}^{c}_{2t} \end{cases}$$

and (73) becomes

$$\sum_{j \in \mathcal{J}_{2t}} \frac{Q_{jt}}{N_t X^0_{jt}} = H_t \left[\chi \sum_{j \in \mathcal{J}^{\chi}_{2t}} n^0_j + \sum_{j \in \mathcal{J}^{c}_{2t}} n^0_j \right] = H_t[\chi \bar{n}^{\chi}_t + \bar{n}^c_t].$$

$$Q_t = \frac{1 - \alpha_1}{1 - n^0_1} H_t[\chi \bar{n}^{\chi}_t + \bar{n}^c_t] + \frac{\alpha_1}{n^0_1} \frac{\hat{X}_{1t}}{\hat{X}^0_{1t}} n_{1t}. \tag{C22}$$

For Examples 9.1-9.4, this means

$$Q_t = \begin{cases} (1 - \alpha_1)[(1 - \kappa) + \kappa\chi] + \alpha_1 \frac{\hat{X}_{1t}}{\hat{X}^0_{1t}} & \text{for } t = t_0 \\ \frac{(1-\alpha_1)}{(1-n^0_1)} H_t \bar{n}_t + \frac{\alpha_1}{n^0_1} \frac{\hat{X}_{1t}}{\hat{X}^0_{1t}} n_{1t} & \text{for } t > t_0 \end{cases}. \tag{C23}$$

For Example 9.5 we have

$$Q_t = \begin{cases} (1 - \alpha_1)[(1 - \kappa) + \kappa\chi] + \alpha_1 \frac{\hat{X}_{1t}}{\hat{X}^0_{1t}} & \text{for } t = t_0 \\ \frac{1-\alpha_1}{1-n^0_1} H_t[\chi \bar{n}^{\chi}_t + \bar{n}^c_t] + \frac{\alpha_1}{n^0_1} \frac{\hat{X}_{1t}}{\hat{X}^0_{1t}} n_{1t} & \text{for } t = t_0 + 1, ..., t_0 + D - 1 \\ \frac{(1-\alpha_1)}{(1-n^0_1)} H_t \bar{n}_t + \frac{\alpha_1}{n^0_1} \frac{\hat{X}_{1t}}{\hat{X}^0_{1t}} n_{1t} & \text{for } t = t_0 + D, t_0 + D + 1, ... \end{cases}.$$

Real GDP for Examples 10.1-10.3. In this case (54) states

$$Q_{jt_0} = \begin{cases} \zeta n_{jt_0} N_{t_0} X^0_{jt_0} & \text{for supply-impacted goods} \\ n_{jt_0} N_{t_0} X^0_{jt_0} & \text{for nonimpacted goods} \end{cases}.$$

for which (73) becomes

$$
\begin{aligned}
Q_{t_0} &= \frac{1-\alpha_1}{1-n_1^0}[(1-\kappa)+\kappa\zeta](1-n_{1t_0}) + \left(\frac{\alpha_1}{n_1^0}\right)\left(\frac{\hat{X}_{1t_0}}{\hat{X}_{1t_0}^0}\right) n_{1t_0} \\
&= (1-\alpha_1)[(1-\kappa)+\kappa\zeta] + \alpha_1 \left(\frac{\hat{X}_{1t_0}}{\hat{X}_{1t_0}^0}\right).
\end{aligned}
$$

Note this is identical to (C23) with χ replaced by ζ.

Appendix D. Bounds on Jensen's Inequality (online)

Proposition D1. *Let*

$$\delta_t = \log\left[\frac{\exp(S_t) - \exp(R_t)}{S_t - R_t}\right] - \left[\frac{S_t + R_t}{2}\right] \tag{D1}$$

where $\delta_t = \delta$ is constant along the steady-state growth path. If

$$\frac{[1 - \beta(1 - k_X)]k_U}{\beta k_\pi} > \delta, \tag{D2}$$

then

$$\alpha_1 + \tau(1 - \alpha_1) < n_1^0. \tag{D3}$$

Proof of Proposition D1.

We first show that $\delta_t = \delta$ is constant along the steady-state growth path:

$$\begin{aligned}
\delta_{t+1} &= \log\left[\frac{[\exp(S_t + g) - \exp(R_t + g)]}{S_t + g - (R_t + g)}\right] - \left[\frac{S_t + g + R_t + g}{2}\right] \\
&= g + \log\left[\frac{\exp(S_t) - \exp(R_t)}{S_t - R_t}\right] - \left[\frac{S_t + R_t}{2}\right] - \frac{2g}{2} \\
&= \delta_t.
\end{aligned}$$

Let $I_t^0 = \sum_{j\in\mathcal{J}_t} P_{jt}^0 Q_{jt}^0 / P_{1t}^0$ denote steady-state real national income. The specialized receive a total share $(1-\alpha_1)(1-\tau)$ and the nonspecialized $\alpha_1 + \tau(1-\alpha_1)$, and thus per capita receive

$$\begin{aligned}
Y_t^0 &= \frac{(1-\alpha_1)(1-\tau)}{1 - n_1^0} I_t^0 \\
\bar{Y}_{1t}^0 &= \frac{\alpha_1 + \tau(1-\alpha_1)}{n_1^0} I_t^0 \\
Y_t^0 - \bar{Y}_{1t}^0 &= \frac{n_1^0 - [\alpha_1 + \tau(1-\alpha_1)]}{n_1^0(1 - n_1^0)} I_t^0
\end{aligned}$$

so (D3) holds whenever $Y_t^0 > \bar{Y}_{1t}^0$. Note that $\bar{Y}_{1t}^0$ could alternatively be calculated as

$$\bar{Y}_{1t}^0 = \int_{\log X_{1t}^*}^{S_t} \frac{\exp(z)dz}{S_t - R_t} + \frac{C_t^0}{P_{1t}^0}\int_{R_t}^{\log X_{1t}^*} \frac{dz}{S_t - R_t}.$$

Expression (37) and Proposition 3 established that

$$h_Y^0 = \log X_{1t}^{*0} - \log(C_t^0/P_{1t}^0) > 0$$

so

$$\bar{Y}^0_{1t} < \int_{\log X^*_{1t}}^{S_t} \frac{\exp(z)dz}{S_t - R_t} + X^*_{1t} \int_{R_t}^{\log X^*_{1t}} \frac{dz}{S_t - R_t} = \tilde{Y}^0_{1t}. \tag{D4}$$

Thus if $Y^0_t > \tilde{Y}^0_{1t}$, then also $Y^0_t > \bar{Y}^0_{1t}$. Thus the proof will be complete if we can show that (D2) implies that $Y^0_t > \tilde{Y}^0_{1t}$.

From (39) and (31),

$$h^0_Y = -k_U + k_\pi \left[\frac{\beta}{1 - \beta(1 - k_X)}\right] \log \tilde{Y}^0 \tag{D5}$$

where from (34), $\log \tilde{Y}^0 = \log Y^0_t - \log \tilde{X}^0_{1t}$. Since $h^0_Y > 0$, (D5) implies

$$\log \tilde{Y}^0 > \frac{[1 - \beta(1 - k_X)]k_U}{\beta k_\pi}.$$

Condition (D2) then establishes that $\log \tilde{Y}^0 > \delta$ meaning $\log Y^0_t > \log \tilde{X}^0_{1t} + \delta$. Thus we will have succeeded in showing that $\log Y^0_t > \log \tilde{Y}^0_{1t}$ if we show that $\log \tilde{Y}^0_{1t} < \log \tilde{X}^0_{1t} + \delta$. From (D4) and (22), this means establishing

$$\log \left[\int_{\log X^*_{1t}}^{S_t} \frac{\exp(z)dz}{S_t - R_t} + X^*_{1t} \int_{R_t}^{\log X^*_{1t}} \frac{dz}{S_t - R_t}\right] < \int_{\log X^*_{1t}}^{S_t} \frac{zdz}{S_t - R_t} + \log X^*_{1t} \int_{R_t}^{\log X^*_{1t}} \frac{dz}{S_t - R_t} + \delta. \tag{D6}$$

For $z^* = \log X^*_{1t}$ define the functions

$$k(z^*) = \int_{z^*}^{S} \frac{\exp(z)dz}{S - R} + \exp(z^*) \int_R^{z^*} \frac{dz}{S - R}$$

$$Q(z^*) = \log[k(z^*)] - \int_{z^*}^{S} \frac{zdz}{S - R} - z^* \int_R^{z^*} \frac{dz}{S - R}$$

whose derivatives are

$$\frac{dk(z^*)}{dz^*} = \frac{-\exp(z^*)}{S - R} + \frac{\exp(z^*)}{S - R} + \exp(z^*) \int_R^{z^*} \frac{dz}{S - R} = \exp(z^*) \int_R^{z^*} \frac{dz}{S - R}$$

$$\begin{aligned}\frac{dQ(z^*)}{dz^*} &= \frac{\exp(z^*)}{k(z^*)} \int_R^{z^*} \frac{dz}{S - R} + \frac{z^*}{S - R} - \frac{z^*}{S - R} - \int_R^{z^*} \frac{dz}{S - R} \\ &= \left[\int_R^{z^*} \frac{dz}{S - R}\right] \left[\frac{\exp(z^*)}{k(z^*)} - 1\right].\end{aligned}$$

Since $k(z^*) \geq \exp(z^*)$, this derivative is negative, meaning this function reaches its maximum at the lowest possible value of z^*, namely $z^* = R$,

$$Q(z^*) \leq Q(R) = \log\left[\int_R^S \frac{\exp(z)dz}{S-R}\right] - \int_R^S \frac{zdz}{S-R} = \log\left[\frac{\exp(S)-\exp(R)}{S-R}\right] - \left[\frac{S+R}{2}\right] = \delta,$$

which is $\log E(x_{it}) - E[\log x_{it}]$ when $\log x_{it} \sim U(R,S)$. From the definition of $Q(\log X_{1t}^{*0})$, this means

$$\log\left[\int_{\log X_{1t}^{*0}}^S \frac{\exp(z)dz}{S-R} + \exp(\log X_{1t}^{*0}) \int_R^{\log X_{1t}^{*0}} \frac{dz}{S-R}\right] - \int_{\log X_{1t}^{*0}}^S \frac{zdz}{S-R} - \log X_{1t}^{*0} \int_R^{\log X_{1t}^{*0}} \frac{dz}{S-R} < \delta$$

establishing (D6).

Note that (D2) is a sufficient, but not a necessary, condition to guarantee (D3). Typically (D3) also holds even when (D2) does not.