Econ 226, Spring 2013

Answers to practice midterm exam

1a.) $N = q$, $n = 1$, $\mathbf{\Lambda} = 1$

b.) Posterior is in the same family of distributions as prior, namely $\mathbf{\Omega}^{-1}|\mathbf{y} \sim W(N^{;*}, \mathbf{\Lambda}^*)$

c.)

$$\begin{aligned} p(\mathbf{\Omega}^{-1}|\mathbf{y}) &\propto |\mathbf{\Omega}|^{-T/2} \exp\left[(-1/2)\text{trace}\left(\mathbf{\Omega}^{-1}\textstyle\sum_{t=1}^T \mathbf{y}_t\mathbf{y}_t'\right)\right] \times \\ & \quad |\mathbf{\Omega}|^{-(N-n-1)/2} \exp\left[(-1/2)\text{trace}\left(\mathbf{\Omega}^{-1}\mathbf{\Lambda}\right)\right] \\ &= |\mathbf{\Omega}|^{-(N^*-n-1)/2} \exp\left[(-1/2)\text{trace}\left(\mathbf{\Omega}^{-1}\mathbf{\Lambda}^*\right)\right] \end{aligned}$$

for $N^* = N + T$, $\mathbf{\Lambda}^* = \mathbf{\Lambda} + \sum_{t=1}^T \mathbf{y}_t\mathbf{y}_t'$

d.) n is as given, $N \to 0$, $\mathbf{\Lambda} \to \mathbf{0}$

2a.) Notice $\mathbf{m}^* = E(\boldsymbol{\beta}|\mathbf{Y})$ and

$$\begin{aligned} E\left[(\boldsymbol{\beta} - \hat{\boldsymbol{\beta}})'\mathbf{W}(\boldsymbol{\beta} - \hat{\boldsymbol{\beta}})|\mathbf{Y}\right] &= E\left[(\boldsymbol{\beta} - \mathbf{m}^* + \mathbf{m}^* - \hat{\boldsymbol{\beta}})'\mathbf{W}(\boldsymbol{\beta} - \mathbf{m}^* + \mathbf{m}^* - \hat{\boldsymbol{\beta}})|\mathbf{Y}\right] \\ &= E\left[(\boldsymbol{\beta} - \mathbf{m}^*)'\mathbf{W}(\boldsymbol{\beta} - \mathbf{m}^*)|\mathbf{Y}\right] + E\left[(\mathbf{m}^* - \hat{\boldsymbol{\beta}})'\mathbf{W}(\mathbf{m}^* - \hat{\boldsymbol{\beta}})|\mathbf{Y}\right] \end{aligned}$$

because

$$\begin{aligned} E\left[(\boldsymbol{\beta} - \mathbf{m}^*)'\mathbf{W}(\mathbf{m}^* - \hat{\boldsymbol{\beta}})|\mathbf{Y}\right] &= E\left[(\boldsymbol{\beta} - \mathbf{m}^*)'|\mathbf{Y}\right]\mathbf{W}(\mathbf{m}^* - \hat{\boldsymbol{\beta}}) \\ &= \mathbf{0}'\mathbf{W}(\mathbf{m}^* - \hat{\boldsymbol{\beta}}) \\ &= 0. \end{aligned}$$

Thus $E\left[(\boldsymbol{\beta} - \hat{\boldsymbol{\beta}})'\mathbf{W}(\boldsymbol{\beta} - \hat{\boldsymbol{\beta}})|\mathbf{Y}\right]$ is minimized by choosing $\hat{\boldsymbol{\beta}} = \mathbf{m}^*$ for any positive-definite matrix $\mathbf{W}$.

b.)

$$\begin{aligned} \mathbf{m}^* &= \left(T^{-1}\mathbf{M}^{-1} + T^{-1}\textstyle\sum \mathbf{x}_t\mathbf{x}_t'\right)^{-1}\left(T^{-1}\mathbf{M}^{-1}\mathbf{m} + T^{-1}\textstyle\sum \mathbf{x}_t y_t\right) \\ &\xrightarrow{p} \left(\text{plim } T^{-1}\textstyle\sum \mathbf{x}_t\mathbf{x}_t'\right)^{-1}\left(\text{plim } T^{-1}\textstyle\sum \mathbf{x}_t y_t\right) \\ &= \boldsymbol{\beta} \end{aligned}$$

c.) Use the Cholesky factorization $\mathbf{M}_T^* = \mathbf{P}_T^*\mathbf{P}_T^{*\prime}$ to write $(\sigma\mathbf{P}_T^*)^{-1}(\mathbf{z}_T - \mathbf{m}^*) \sim N(\mathbf{0}, \mathbf{I}_k)$. Notice

$$\begin{aligned} T\mathbf{M}_T^* &= T\left(\mathbf{M}^{-1} + \textstyle\sum \mathbf{x}_t\mathbf{x}_t'\right)^{-1} \\ &= \left(T^{-1}\mathbf{M}^{-1} + T^{-1}\textstyle\sum \mathbf{x}_t\mathbf{x}_t'\right)^{-1} \\ &\xrightarrow{p} \mathbf{Q}^{-1}. \end{aligned}$$

Hence $T^{1/2}\mathbf{P}_T^* \xrightarrow{p} \mathbf{P}$ for $\mathbf{PP}' = \mathbf{Q}^{-1}$. Thus

$$\begin{aligned}(\sigma\mathbf{P}_T^*)^{-1}(\mathbf{z}_T - \mathbf{m}^*) - T^{1/2}(\sigma\mathbf{P})^{-1}(\mathbf{z}_T - \mathbf{m}^*) &= T^{1/2}\left[(\sigma T^{1/2}\mathbf{P}_T^*)^{-1} - (\sigma\mathbf{P})^{-1}\right](\mathbf{z}_T - \mathbf{m}) \\ &\xrightarrow{p} \mathbf{0}.\end{aligned}$$

Since $(\sigma\mathbf{P}_T^*)^{-1}(\mathbf{z}_T - \mathbf{m}^*) \sim N(\mathbf{0}, \mathbf{I}_k)$, it follows that $T^{1/2}(\sigma\mathbf{P})^{-1}(\mathbf{z}_T - \mathbf{m}^*) \xrightarrow{L} N(\mathbf{0}, \mathbf{I}_k)$ or

$$T^{1/2}(\mathbf{z}_T - \mathbf{m}^*) \xrightarrow{L} N(\mathbf{0}, \sigma^2\mathbf{PP}') \sim N(\mathbf{0}, \sigma^2\mathbf{Q}^{-1}).$$

In other words, the Bayesian posterior distribution becomes a better and better approximation to the asymptotic distribution of the MLE as T gets larger.