Econ 226, Spring 2013

Practice midterm exam

DIRECTIONS: No books or notes of any kind are allowed. Answer all questions on separate paper. 100 points are possible

1.) (60 points total) Suppose we have an i.i.d. sample of size T on an $(n \times 1)$ vector $\mathbf{y}_t$ that is Gaussian with known mean $\mathbf{0}$ and unknown $(n \times n)$ variance-covariance matrix $\boldsymbol{\Omega}$, so that the sample likelihood is

$$p(\mathbf{Y}|\boldsymbol{\Omega}) = \frac{1}{(2\pi)^{nT/2}}|\boldsymbol{\Omega}|^{-T/2} \exp\left(-(1/2)\textstyle\sum_{t=1}^{T} \mathbf{y}_t'\boldsymbol{\Omega}^{-1}\mathbf{y}_t\right).$$

For the prior distribution we use a Wishart distribution for the inverse of the variance-covariance matrix:

$$\begin{aligned} p(\boldsymbol{\Omega}^{-1}) &= \left[2^{Nn/2}\pi^{n(n-1)/4}\prod_{j=1}^{n}\Gamma\left(\frac{N+1-j}{2}\right)\right]^{-1} \times \\ & \quad |\boldsymbol{\Lambda}|^{N/2}|\boldsymbol{\Omega}|^{-(N-n-1)/2} \exp\left[-\frac{1}{2}\text{trace}\left(\boldsymbol{\Omega}^{-1}\boldsymbol{\Lambda}\right)\right] \end{aligned}$$

a.) (10 points) What values would you use for N, n, and $\boldsymbol{\Lambda}$ to have the Wishart distribution represent a $\chi^2(q)$ distribution as a special case?

b.) (10 points) The Wishart distribution is the natural conjugate prior for this likelihood function. What does that mean?

c.) (30 points) Find the posterior distribution $p(\boldsymbol{\Omega}^{-1}|\mathbf{Y})$. Hint: use the fact that if $\mathbf{A}$ is $(n \times r)$ and $\mathbf{B}$ is $(r \times n)$, then $\text{trace}(\mathbf{AB}) = \text{trace}(\mathbf{BA})$.

d.) (10 points) What values would you use for n, N, and $\boldsymbol{\Lambda}$ to represent a diffuse prior?

2.) (40 points total) Consider a Gaussian regression model $y_t = \mathbf{x}_t'\boldsymbol{\beta} + \varepsilon_t$ for deterministic regressors $\mathbf{x}_t$ with i.i.d. residuals $\varepsilon_t \sim N(0, \sigma^2)$ with σ^2 known and $\boldsymbol{\beta}$ a $(k \times 1)$ vector. Suppose we have a Bayesian prior for the coefficient vector of the form $\boldsymbol{\beta} \sim N(\mathbf{m}, \sigma^2\mathbf{M})$. You may find the following results helpful in answering this question. The OLS coefficient estimator is given by $\mathbf{b} = \left(\sum_{t=1}^{T}\mathbf{x}_t\mathbf{x}_t'\right)^{-1}\left(\sum_{t=1}^{T}\mathbf{x}_t y_t\right)$ and has classical asymptotic distribution characterized by $\sqrt{T}(\mathbf{b} - \boldsymbol{\beta}) \xrightarrow{L} N(\mathbf{0}, \sigma^2\mathbf{Q}^{-1})$ for $\mathbf{Q} = \text{plim } T^{-1}\sum_{t=1}^{T}\mathbf{x}_t\mathbf{x}_t'$. The log likelihood function is

$$\log p(\mathbf{y}|\boldsymbol{\beta}; \sigma) = -(T/2)\log(2\pi\sigma^2) - \sum_{t=1}^{T}\frac{(y_t - \mathbf{x}_t'\boldsymbol{\beta})^2}{2\sigma^2}.$$

The Bayesian posterior distribution is $\boldsymbol{\beta}|\mathbf{y} \sim N(\mathbf{m}^*, \sigma^2\mathbf{M}^*)$ for

$$\mathbf{M}^* = \left(\mathbf{M}^{-1} + \sum_{t=1}^{T} \mathbf{x}_t\mathbf{x}_t^{'}\right)^{-1}$$

$$\mathbf{m}^* = \mathbf{M}^* \left(\mathbf{M}^{-1}\mathbf{m} + \sum_{t=1}^{T} \mathbf{x}_t y_t\right).$$

a.) (20 points) Suppose you have a quadratic loss function, that is, if you announce your estimate to be $\hat{\boldsymbol{\beta}}$ and the true value turns out to be $\boldsymbol{\beta}$, then your loss is $\ell(\boldsymbol{\beta}, \hat{\boldsymbol{\beta}}) = (\boldsymbol{\beta} - \hat{\boldsymbol{\beta}})'\mathbf{W}(\boldsymbol{\beta} - \hat{\boldsymbol{\beta}})$ for a given positive-definite weighting matrix $\mathbf{W}$ that characterizes your preferences. Calculate the optimal Bayesian estimate $\hat{\boldsymbol{\beta}}$ given the data.

b.) (10 points) From a classical perspective, the variable $\mathbf{m}^*$ is a function of the data with some asymptotic behavior as the sample size gets large. Calculate the plim of $\mathbf{m}^*$ as $T \to \infty$ treating $\mathbf{m}$ and $\mathbf{M}$ as fixed parameters.

c.) (10 points) From a classical perspective the distribution $N(\mathbf{m}^*, \sigma^2\mathbf{M}^*)$ is also a function of the data. If $\mathbf{z}_T \sim N(\mathbf{m}^*, \sigma^2\mathbf{M}^*)$, calculate from a classical perspective the asymptotic distribution of $\mathbf{z}_T$ as $T \to \infty$.