Econ 226, Spring 2013

Answers to midterm exam

1a.)

$$\mathcal{L} = -(T/2)\log(2\pi) - (T/2)\log\sigma^2 - \frac{\sum y_t^2}{2\sigma^2}$$

$$\frac{\partial\mathcal{L}}{\partial\sigma^2} = \frac{-T}{2\sigma^2} + \frac{\sum y_t^2}{2\sigma^4} = 0$$

$$\hat{\sigma}^2 = T^{-1}\sum y_t^2$$

b.)

$$p(\mathbf{y}|\sigma) = \frac{1}{(2\pi\sigma^2)^{T/2}}\exp\left(-\frac{T\hat{\sigma}^2}{2\sigma^2}\right)$$

depends on data $\mathbf{y}$ only through $\hat{\sigma}^2$.

c.) If $\lambda = N$ then $E(\sigma^{-2}) = 1$. For increasing confidence make both λ and N larger while keeping $\lambda/N = 1$

d.)

$$\begin{aligned} p(\sigma^{-2}|\mathbf{y}) &\propto \frac{1}{(\sigma^2)^{T/2}}\exp\left(-\frac{\sum y_t^2}{2\sigma^2}\right)\frac{1}{(\sigma^2)^{[(N/2)-1]}}\exp\left(-\frac{\lambda}{2\sigma^2}\right) \\ &= \frac{1}{(\sigma^2)^{[(N^*/2)-1]}}\exp\left(-\frac{\lambda^*}{2\sigma^2}\right) \end{aligned}$$

for $N^* = N + T$ and $\lambda^* = \lambda + \sum y_t^2$

e.)

$$E(\sigma^{-2}|\mathbf{y}) = \frac{N^*}{\lambda^*} = \frac{N+T}{\lambda + T\hat{\sigma}^2} = \frac{(N/T)+1}{(\lambda/T)+\hat{\sigma}^2} \xrightarrow{p} \frac{1}{\sigma^2}$$

since $\hat{\sigma}^2 \xrightarrow{p} \sigma^2$.

$$Var(\sigma^{-2}|\mathbf{y}) = \frac{2N^*}{(\lambda^*)^2} = \frac{(2N/T^2)+(2/T)}{[(\lambda/T)+\hat{\sigma}^2]^2} \xrightarrow{p} 0$$

2a.) $\mathbf{M}^{-1} \to \mathbf{0}$, $\mathbf{M}^* \to \left(\sum_{t=1}^{T} \mathbf{x}_t \mathbf{x}_t^{'}\right)^{-1}$, $\mathbf{m}^* \to \mathbf{b}$

b.) Calculate the Cholesky factor $\mathbf{PP}' = \mathbf{M}$ and let $\tilde{T} = T + k$, $\tilde{y}_t = y_t$ for $t = 1, ..., T$, $\tilde{\mathbf{x}}_t = \mathbf{x}_t$ for $t = 1, ..., T$, and

$$\begin{bmatrix} \tilde{y}_{T+1} \\ \vdots \\ \tilde{y}_{T+k} \end{bmatrix} = \mathbf{P}^{-1}\mathbf{m}$$

$$\begin{bmatrix} \tilde{\mathbf{x}}_{T+1} \\ \vdots \\ \tilde{\mathbf{x}}_{T+k} \end{bmatrix} = \mathbf{P}^{-1}$$

Then

$$\begin{aligned} \sum_{\tilde{t}=1}^{\tilde{T}} \tilde{\mathbf{x}}_{\tilde{t}} \tilde{\mathbf{x}}_{\tilde{t}}^{'} &= \sum_{t=1}^{T} \mathbf{x}_t \mathbf{x}_t^{'} + \mathbf{P}^{-1\prime}\mathbf{P}^{-1} = \sum_{t=1}^{T} \mathbf{x}_t \mathbf{x}_t^{'} + \mathbf{M}^{-1} \\ \sum_{\tilde{t}=1}^{\tilde{T}} \tilde{\mathbf{x}}_{\tilde{t}} \tilde{y}_{\tilde{t}} &= \sum_{t=1}^{T} \mathbf{x}_t y_t + \mathbf{P}^{-1\prime}\mathbf{P}^{-1}\mathbf{m} = \sum_{t=1}^{T} \mathbf{x}_t y_t + \mathbf{M}^{-1}\mathbf{m} \end{aligned}$$