Econ 226, Spring 2013

Midterm Exam

DIRECTIONS: No books or notes of any kind are allowed. Answer all questions on separate paper. 100 points are possible

1.) (70 points total) This question calls for making use of a gamma distribution, for which you might find the following results useful. A variable x is said to have a gamma distribution with parameters N and λ, denoted $x \sim G(N, \lambda)$, if it has density

$$p(x) = \begin{cases} \frac{(\lambda/2)^{N/2}}{\Gamma(N/2)} x^{[(N/2)-1]} \exp(-\lambda x/2) & \text{if } x > 0 \\ 0 & \text{otherwise} \end{cases}$$

for $\Gamma(.)$ the gamma function. If $x \sim G(N, \lambda)$, then $E(x) = N/\lambda$ and $Var(x) = 2N/\lambda^2$. If $x \sim \chi^2(N)$, then $x/\lambda \sim G(N, \lambda)$.

For this question we are concerned with inference about the variance σ^2 from an i.i.d. Gaussian sample known to have mean zero: $y_t \sim N(0, \sigma^2)$, for which the likelihood function for the full set of observed data $\mathbf{y} = (y_1, ..., y_T)'$ is

$$p(\mathbf{y}|\sigma) = \frac{1}{(2\pi\sigma^2)^{T/2}} \exp\left\{ \frac{-\sum_{t=1}^{T} y_t^2}{2\sigma^2} \right\}$$

a.) (10 points) Show that the classical maximum likelihood estimate is given by $\hat{\sigma}^2 = T^{-1}\sum_{t=1}^{T} y_t^2$.

b.) (10 points) Show that $\hat{\sigma}^2$ is a sufficient statistic.

c.) (10 points) Suppose that our prior distribution for the reciprocal of σ^2 is taken to be a gamma distribution: $\sigma^{-2} \sim G(N, \lambda)$. How would you represent a prior expectation that $E(\sigma^{-2}) = 1$? How would you represent increasing confidence in the belief that σ^{-2} is close to 1?

d.) (15 points) Show that the posterior distribution is $\sigma^{-2}|\mathbf{y} \sim G(N^*, \lambda^*)$. Find the values for N^* and λ^*.

e.) (25 points) Calculate the posterior mean $E(\sigma^{-2}|\mathbf{y})$ and variance $Var(\sigma^{-2}|\mathbf{y})$. Show that from a classical perspective, $E(\sigma^{-2}|\mathbf{y})$ converges in probability to the true value σ^{-2} as $T \to \infty$ and $Var(\sigma^{-2}|\mathbf{y}) \xrightarrow{p} 0$.

2.) (30 points total) Consider a Gaussian regression model $y_t = \mathbf{x}_t'\boldsymbol{\beta} + \varepsilon_t$ for deterministic regressors $\mathbf{x}_t$ with i.i.d. residuals $\varepsilon_t \sim N(0, \sigma^2)$ with σ^2 known and $\boldsymbol{\beta}$ a $(k \times 1)$ vector. Suppose we have a Bayesian prior for the coefficient vector of the form $\boldsymbol{\beta} \sim N(\mathbf{m}, \sigma^2\mathbf{M})$. You may find the following results helpful in answering this question. The OLS coefficient estimator is given by $\mathbf{b} = \left(\sum_{t=1}^T \mathbf{x}_t\mathbf{x}_t'\right)^{-1}\left(\sum_{t=1}^T \mathbf{x}_t y_t\right)$ and has classical asymptotic distribution characterized by $\sqrt{T}(\mathbf{b} - \boldsymbol{\beta}) \xrightarrow{L} N(\mathbf{0}, \sigma^2\mathbf{Q}^{-1})$ for $\mathbf{Q} = \text{plim } T^{-1}\sum_{t=1}^T \mathbf{x}_t\mathbf{x}_t'$. The log likelihood function is

$$\log p(\mathbf{y}|\boldsymbol{\beta};\sigma) = -(T/2)\log(2\pi\sigma^2) - \sum_{t=1}^T \frac{(y_t - \mathbf{x}_t'\boldsymbol{\beta})^2}{2\sigma^2}.$$

The Bayesian posterior distribution is $\boldsymbol{\beta}|\mathbf{y} \sim N(\mathbf{m}^*, \sigma^2\mathbf{M}^*)$ for

$$\mathbf{M}^* = \left(\mathbf{M}^{-1} + \sum_{t=1}^T \mathbf{x}_t\mathbf{x}_t'\right)^{-1}$$

$$\mathbf{m}^* = \mathbf{M}^*\left(\mathbf{M}^{-1}\mathbf{m} + \sum_{t=1}^T \mathbf{x}_t y_t\right).$$

a.) (15 points) How would you represent a diffuse prior for this example? Calculate the values of $\mathbf{M}^*$ and $\mathbf{m}^*$ under your proposed diffuse prior.

b.) (15 points) Suppose you only have available a regression package for which you input $\tilde{T}$ observations on the dependent variable $\{\tilde{y}_1, ..., \tilde{y}_{\tilde{T}}\}$ and $\tilde{T}$ observations on the explanatory variables $\{\tilde{\mathbf{x}}_1, ..., \tilde{\mathbf{x}}_{\tilde{T}}\}$ and the software calculates for you the regression estimate and its variance-covariance matrix

$$\tilde{\mathbf{b}} = \left(\sum_{\tilde{t}=1}^{\tilde{T}} \tilde{\mathbf{x}}_{\tilde{t}}\tilde{\mathbf{x}}_{\tilde{t}}'\right)^{-1}\left(\sum_{\tilde{t}=1}^{\tilde{T}} \tilde{\mathbf{x}}_{\tilde{t}}\tilde{y}_{\tilde{t}}\right)$$

$$\tilde{\mathbf{V}} = \sigma^2\left(\sum_{\tilde{t}=1}^{\tilde{T}} \tilde{\mathbf{x}}_{\tilde{t}}\tilde{\mathbf{x}}_{\tilde{t}}'\right)^{-1}.$$

What values would you use for $\tilde{y}_{\tilde{t}}$, $\tilde{\mathbf{x}}_{\tilde{t}}$, and $\tilde{T}$ in order to use the regression package to give you the posterior distribution, that is, in order to have $\tilde{\mathbf{b}} = \mathbf{m}^*$ and $\tilde{\mathbf{V}} = \sigma^2\mathbf{M}^*$? Hint: for partial credit you can give the answer for the special case when $\mathbf{M}$ is diagonal.