Paper due Wednesday June 3

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VI. Time-varying variances

A. Overview

y_t = return on a stock in period t

μ = population mean return

$y_t = \mu + u_t$

Observation: u_t is almost impossible to predict

$$E(u_t | u_{t-1}, u_{t-2}, \ldots) = 0$$

However: u_t^2 does seem to be quite forecastable

Question 1: how should we forecast u_t^2?

One answer: autoregression on its own lagged values:

$$u_t^2 = \varsigma + \alpha_1 u_{t-1}^2 + \alpha_2 u_{t-2}^2 + \cdots + \alpha_m u_{t-m}^2 + w_t$$

$E(w_t) = 0$

$E(w_t^2) = \lambda^2$

$E(w_t w_\tau) = 0$ if $t \neq \tau$

Question 2: what kind of data-generating process would imply such a forecast?

$u_t = \sqrt{h_t}\, \varepsilon_t$

$\varepsilon_t \sim$ i.i.d. $(0,1)$ (e.g. $N(0,1)$)

$h_t = \varsigma + \alpha_1 u_{t-1}^2 + \alpha_2 u_{t-2}^2 + \cdots + \alpha_m u_{t-m}^2$

Definition: a regression model with Gaussian $ARCH(m)$ error is characterized by

$y_t = \mathbf{x}_t' \boldsymbol{\beta} + u_t$

$u_t = \sqrt{h_t}\, v_t$

$v_t \sim$ i.i.d. $N(0,1)$

$h_t = \varsigma + \alpha_1 u_{t-1}^2 + \alpha_2 u_{t-2}^2 + \cdots + \alpha_m u_{t-m}^2$

ARCH = autoregressive conditional heteroskedasticity

Note: even though u_t has a distribution that is conditionally Gaussian,

$$u_t | u_{t-1}, u_{t-2} \sim N(0, h_t),$$

its unconditional distribution is non-Gaussian (fatter tails)

parameters of Gaussian $ARCH(m)$ regression: $\boldsymbol{\theta} = (\boldsymbol{\beta}', \boldsymbol{\alpha}', \varsigma)'$

estimate by maximum likelihood:

$$\Omega_{t-1} = \mathbf{x}_t, y_{t-1}, \mathbf{x}_{t-1}, y_{t-2}, \mathbf{x}_{t-2}, \ldots$$

$$y_t | \Omega_{t-1} \sim N(\mathbf{x}_t' \boldsymbol{\beta}, h_t)$$

$$h_t = \varsigma + \alpha_1 u_{t-1}^2 + \alpha_2 u_{t-2}^2 + \cdots + \alpha_m u_{t-m}^2$$

$$u_t = y_t - \mathbf{x}_t' \boldsymbol{\beta}$$

$$f(y_t | \Omega_{t-1}; \boldsymbol{\theta}) = \frac{1}{\sqrt{2\pi h_t}} \exp\left[-\frac{(y_t - \mathbf{x}_t' \boldsymbol{\beta})^2}{2h_t} \right]$$

$$\mathcal{L}(\boldsymbol{\theta}) = \sum_{t=1}^{T} \log f(y_t | \Omega_{t-1}; \boldsymbol{\theta})$$

choose $\boldsymbol{\theta}$ numerically to maximize $\mathcal{L}(\boldsymbol{\theta})$ subject to $\varsigma \geq 0,\ \alpha_j \geq 0$ (e.g., set $\alpha_j = \lambda_j^2$)

use first m values of y_t and $\mathbf{x}_t$ for conditioning

Although a Gaussian specification for v_t is natural starting point, stock returns are better modeled using a Student t

$y_t|\Omega_{t-1} \sim$ Student t with $\nu > 2$ degrees of freedom

conditional mean:

$$E(y_t|\Omega_{t-1}) = \mathbf{x}_t'\boldsymbol{\beta}$$

conditional variance:

$$E[(y_t - \mathbf{x}_t'\boldsymbol{\beta})^2|\Omega_{t-1}] = h_t$$

$$\log f(y_t|\Omega_{t-1};\boldsymbol{\theta}) =$$

$$\log\left\{\frac{\Gamma[(v+1)/2]}{\sqrt{\pi}\,\Gamma(v/2)}(v-2)^{-1/2}\right\} - \frac{1}{2}\log(h_t)$$

$$-\left[\frac{(v+1)}{2}\right]\log\left[1+\frac{(y_t-\mathbf{x}_t'\boldsymbol{\beta})^2}{h_t(v-2)}\right]$$

$$h_t = \varsigma + \alpha_1 u_{t-1}^2 + \alpha_2 u_{t-2}^2 + \cdots + \alpha_m u_{t-m}^2$$

Issues:

(1) covariance-stationary if

$1-\alpha_1 z - \cdots - \alpha_m z^m = 0$

implies that $\|z\| > 1$

(2) $E(u_t^2|u_{t-1},\ldots,u_{t-m}) > 0$

Sufficient conditions:

$\varsigma > 0$

$\alpha_j \geq 0 \quad j = 1,\ldots,m$

$\alpha_1 + \alpha_2 + \cdots + \alpha_m < 1$

generalized autoregressive conditional heteroskedasticity ($GARCH$) Tim Bollerslev dissertation

$$u_t = \sqrt{h_t}\, v_t$$
$$v_t \sim (0,1)$$

$$u_t = \sqrt{h_t}\, v_t$$
$$v_t \sim (0,1)$$

$ARCH(m)$:

$$h_t = \varsigma + \alpha(L)u_t^2$$
$$\alpha(L) = \alpha_1 L + \alpha_2 L^2 + \cdots \alpha_m L^m$$

$ARCH(\infty)$:

$$h_t = \varsigma + \pi(L)u_t^2$$
$$\pi(L) = \sum_{j=0}^{\infty} \pi_j L^j$$

parsimony:

$$\pi(L) = \frac{\alpha_1 L + \alpha_2 L^2 + \cdots \alpha_m L^m}{1 - \delta_1 L - \delta_2 L^2 - \cdots - \delta_r L^r}$$

$$
\begin{aligned}
&(1-\delta_1 L-\delta_2 L^2-\cdots-\delta_r L^r)h_t \\
&= (1-\delta_1-\delta_2-\cdots-\delta_r)\varsigma \\
&+(\alpha_1 L+\alpha_2 L^2+\cdots\alpha_m L^m)u_t^2 \\
&u_t \sim GARCH(r,m)
\end{aligned}
$$

almost all applications

use $GARCH(1,1)$

$$(1-\delta_1 L)h_t = \kappa+\alpha_1 L u_t^2$$

$$h_t = \kappa+\delta_1 h_{t-1}+\alpha_1 u_{t-1}^2$$

$$h_t = \kappa+\delta_1 h_{t-1}+\alpha_1 u_{t-1}^2$$

add u_t^2 to both sides:

$$
\begin{aligned}
h_t+u_t^2 &= \kappa+\delta_1 u_{t-1}^2-\delta_1(u_{t-1}^2-h_{t-1}) \\
&\quad +\alpha_1 u_{t-1}^2+u_t^2 \\
u_t^2 &= \kappa+(\delta_1+\alpha_1)u_{t-1}^2+(u_t^2-h_t) \\
&\quad -\delta_1(u_{t-1}^2-h_{t-1}) \\
E(u_t^2|u_{t-1},u_{t-2}\ldots) &= h_t \\
w_t &= u_t^2-h_t \\
u_t^2 &= \kappa+(\delta_1+\alpha_1)u_{t-1}^2+w_t-\delta_1 w_{t-1}
\end{aligned}
$$

$u_t^2 = \kappa + (\delta_1 + \alpha_1)u_{t-1}^2 + w_t - \delta_1 w_{t-1}$

conclusion:

$u_t \sim GARCH(1,1)$

$\Rightarrow u_t^2 \sim ARMA(1,1)$

AR coefficient $= \delta_1 + \alpha_1$

MA coefficient $= -\delta_1$

stationarity requires:

$|\alpha_1 + \delta_1| < 1$

more generally:

$u_t \sim GARCH(r,m)$

$\Rightarrow u_t^2 \sim ARMA(\max\{r,m\},r)$

Why does the conditional variance matter?

1) knowing variance of returns is important for

a) assessing risk

b) portfolio choice

c) options pricing

2) even if you're interested in mean only, correctly modeling the variance could matter for

- a) more accurate hypothesis tests
- b) more efficient estimates

Hamilton, "Macroeconomics and ARCH"

$$
\begin{aligned}
y_t &= \beta_0 + \beta_1 y_{t-1} + u_t \\
u_t &\sim \text{GARCH}(1,1) \\
u_t &= \sqrt{h_t}\, v_t \\
h_t &= \kappa + \alpha u_{t-1}^2 + \delta h_{t-1} \\
v_t &\sim \text{i.i.d. } N(0,1)
\end{aligned}
$$

$y_t = \phi y_{t-1} + u_t$

Usual asymptotics:

$$\sqrt{T}(\hat{\phi} - \phi) = \frac{T^{-1/2} \sum_{t=1}^{T} y_{t-1} u_t}{T^{-1} \sum_{t=1}^{T} y_{t-1}^2}$$

$$E(y_{t-1} u_t)^2 = E(y_{t-1}^2) E(u_t^2)$$

$$T^{-1/2} \sum_{t=1}^{T} y_{t-1} u_t \xrightarrow{L} N(0, E(y_{t-1}^2) E(u_t^2))$$

$$T^{-1} \sum_{t=1}^{T} y_{t-1}^2 \xrightarrow{p} E(y_{t-1}^2)$$

$$\sqrt{T}(\hat{\phi} - \phi) \xrightarrow{L} N(0, E(u_t^2)/E(y_{t-1}^2))$$

$$\sqrt{T}(\hat{\phi}-\phi) \xrightarrow{L} N(0, E(u_t^2)/E(y_{t-1}^2))$$

$$\hat{\sigma}_{\hat{\phi}}^2 = s^2 / \sum_{t=1}^{T} y_{t-1}^2$$

$$T\hat{\sigma}_{\hat{\phi}}^2 \xrightarrow{p} E(u_t^2)/E(y_{t-1}^2)$$

t stat $\xrightarrow{L} N(0,1)$

However, suppose true $\phi = 0$

(so $y_t = u_t$) and $u_t \sim$ GARCH$(1,1)$

$$E(y_{t-1}u_t)^2 = E(u_{t-1}^2 u_t^2)$$

$$= \rho\left\{E(u_t^4) - [E(u_t^2)]^2\right\} + [E(u_t^2)]^2$$

$$\rho = \frac{[1-(\alpha+\delta)\delta]\alpha}{1+\delta^2-2(\alpha+\delta)\delta}$$

If $\alpha = \delta = 0$ (no GARCH), then $\rho = 0$

$$E(u_{t-1}^2 u_t^2) = E(u_{t-1}^2)E(u_t^2)$$

But with GARCH,

$$E(u_{t-1}^2 u_t^2) > E(u_{t-1}^2)E(u_t^2)$$

t stat $\xrightarrow{L} N(0, V_{11})$

$$V_{11} \geq 1$$

$V_{11} \xrightarrow{p} \infty$ as

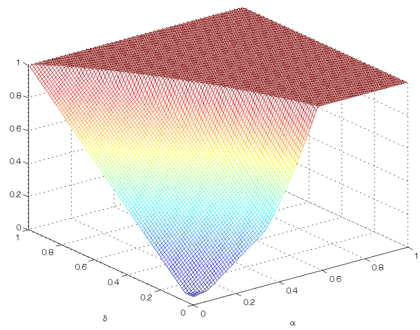
$$3\alpha^2 + 2\alpha\delta + \delta^2 \xrightarrow{p} 1$$

True size of usual t test > 0.05

As fourth moments become infinite, true size $\rightarrow 1$

All t tests reject the true null hypothesis asymptotically with prob 1

Asymptotic rejection probability for OLS *t*-test that autoregressive coefficient is zero as a function of GARCH(1,1) parameters α and δ. Note: null hypothesis is actually true and test has nominal size of 5%.



Taylor rule:

$$\Delta r_t = \gamma_0 + \gamma_1 \pi_t + \gamma_2 y_t + \gamma_3 y_{t-1} + \gamma_r r_{t-1} + \gamma_5 \Delta r_{t-1} + v_t$$

r_t = fed funds rate for quarter t

π_t = inflation

y_t = deviation of real GDP from potential

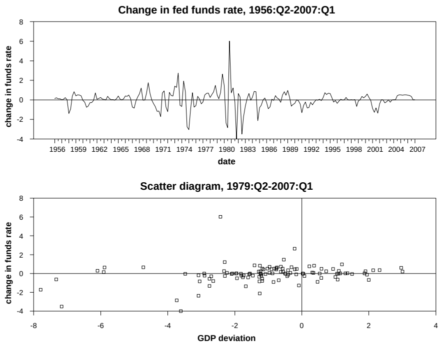
Claim: γ_1 and γ_2 are higher now than in 1970s, which contributes to greater economic stability

Taylor Rule with separate pre- and post-Volcker parameters as estimated by OLS regression (d_t = 1 for t > 1979:Q2).

Regressor	Coefficient	Std error (OLS)	Std error (White)
constant	0.37	0.19	0.19
π_t	0.17	0.07	0.04
y_t	0.18	0.08	0.07
y_{t-1}	-0.07	0.08	0.07
r_{t-1}	-0.21	0.07	0.06
Δr_{t-1}	0.42	0.11	0.13
d_t	-0.50	0.24	0.30
$d_t \pi_t$	0.26	0.09	0.16
$d_t y_t$	0.64	0.14	0.24
$d_t y_{t-1}$	-0.55	0.14	0.21
$d_t r_{t-1}$	0.05	0.08	0.08
$d_t \Delta r_{t-1}$	-0.53	0.13	0.24

Taylor Rule with separate pre- and post-Volcker parameters as estimated by GARCH-t maximum likelihood (d_t = 1 for t > 1979:Q2).

Regressor	Coefficient	Asymptotic std error
constant	0.13	0.08
π_t	0.06	0.03
y_t	0.14	0.03
y_{t-1}	-0.12	0.03
r_{t-1}	-0.07	0.03
Δr_{t-1}	0.47	0.09
d_t	-0.03	0.12
$d_t \pi_t$	0.09	0.04
$d_t y_t$	0.05	0.07
$d_t y_{t-1}$	0.02	0.07
$d_t r_{t-1}$	-0.01	0.03
$d_t \Delta r_{t-1}$	-0.01	0.11



VI. Time-varying variances

A. Overview
B. Extensioins

1. Exponential GARCH (EGARCH)

$u_t = \sqrt{h_t}\, v_t$

$\log h_t = \varsigma + \sum_{j=1}^{\infty} \pi_j [|v_{t-j}| - E|v_{t-j}| + \chi v_{t-j}]$

$v_t \sim$ i.i.d. $(0,1)$

$\pi_j > 0 \Rightarrow$ if $|v_{t-j}| \uparrow$, then $h_t \uparrow$

$\chi = 0 \Rightarrow$ positive v_{t-j} and negative v_{t-j} has identical effects on variance

$$\log h_t = \varsigma + \sum_{j=1}^{\infty} \pi_j [|v_{t-j}| - E|v_{t-j}| + \chi v_{t-j}]$$

$\chi < 0 \Rightarrow$ a decrease in stock price increases variance more than an increase in stock prices (called “leverage effect”)

parsimony:

$$\pi(L) = \frac{\alpha(L)}{1 - \delta(L)}$$

EGARCH(1,1):

$$\log h_t = \kappa + \delta_1 \log h_{t-1} + \alpha_1 \{|v_{t-1}| - E|v_{t-1}| + \chi v_{t-1}\}$$

Nelson proposed generalized error distribution (GED) for v_t

$f(v_t;\eta) = c_\eta \exp\{-(1/2)|v_t/\lambda_\eta|^\eta\}$

where c_η and λ_η are constants to make the density integrate to 1 and have unit variance

$f(v_t;\eta) = c_\eta \exp\{-(1/2)|v_t/\lambda_\eta|^\eta\}$

$\eta = 2 \Rightarrow$

$f(v_t;\eta = 2) = c_2 \exp\{-(1/2)v_t^2/\lambda_2\}$

$\sim N(0,1)$

$\eta = 1 \Rightarrow$ double exponential

$\eta < 2 \Rightarrow$ fatter tails than Normal

$\eta > 2 \Rightarrow$ thinner tails than Normal

2. Realized volatility

Consider continuous-time process:

$p(t) = \mu t + \sigma W(t)$

$W(t) \sim$ standard Brownian motion

e.g., $p(t) =$ log of asset price at t

$p(t) - p(t-h) \sim N(\mu h, \sigma^2 h)$

Divide interval $[t - h, t]$ into n
segments each of length $\Delta = h/n$
segment i starts at $t - h + (i - 1)\Delta$
and ends at $t - h + i\Delta$
segment $i = 1$: $[t - h, t - h + \Delta]$
segment $i = n$: $[t - \Delta, t]$

r_i = return over segment i
$= p(t - h + i\Delta) - p(t - h + (i - 1)\Delta)$
$\sim N(\mu\Delta, \sigma^2\Delta)$

Question 1: Can we get better inference about μ by dividing fixed interval $[t - h, t]$ into smaller segments, that is, by making n bigger?
Answer: no

$\hat{\mu}_n = n^{-1} \sum_{i=1}^{n} r_i \Delta^{-1}$

Recall $r_i \sim N(\mu\Delta, \sigma^2\Delta)$ and $\Delta = h/n$

$\hat{\mu}_n = h^{-1} \sum_{i=1}^{n} [p(t-h+i\Delta) -$

$p(t-h+(i-1)\Delta)]$

$= h^{-1}[p(t) - p(t-h)]$

same estimate regardless of n

$\hat{\mu}_n \sim N(\mu, \sigma^2/h)$

unbiased but not consistent as

$n \to \infty$

To get better estimate, need longer time period (bigger h) not more observations for fixed period (bigger n)

Question 2: Can we get better inference about σ^2 by dividing fixed interval $[t-h, t]$ into smaller segments, that is, by making n bigger?

Answer: yes

Recall $r_i \sim N(\mu\Delta, \sigma^2\Delta)$ and $\Delta = h/n$

$\hat{\sigma}_n^2 = n^{-1}\sum_{i=1}^{n} r_i^2 \Delta^{-1} = h^{-1}\sum_{i=1}^{n} r_i^2$

$\hat{\sigma}_n^2 = h^{-1}\sum_{i=1}^{n}[p(t-h+i\Delta) - p(t-h+(i-1)\Delta)]^2$

$= h^{-1}\sum_{i=1}^{n}(\mu\Delta + \sigma\sqrt{\Delta}\, x_i)^2$

$x_i \sim$ i.i.d. $N(0,1)$

$\hat{\sigma}_n^2 = h^{-1}\sum_{i=1}^{n}(\mu^2\Delta^2 + 2\mu\sigma\Delta^{3/2}x_i + \sigma^2\Delta x_i^2)$

As $n \to \infty$,

$h^{-1}\sum_{i=1}^{n}\mu^2\Delta^2 = h^{-1}n\Delta^2\mu^2$

$= (h/n)\mu^2 \to 0$

$h^{-1}\sum_{i=1}^{n} 2\mu\sigma\Delta^{3/2}x_i = 2\mu\sigma(h/n)^{1/2}n^{-1}\sum_{i=1}^{n} x_i$

$\xrightarrow{p} 0$

$h^{-1}\sum_{i=1}^{n}\sigma^2\Delta x_i^2 = \sigma^2 n^{-1}\sum_{i=1}^{n}\Delta x_i^2 \xrightarrow{p} \sigma^2$

Conclusion:

$\hat{\sigma}_n^2 \xrightarrow{p} \sigma^2$ as $n \to \infty$ for any h

More generally, if

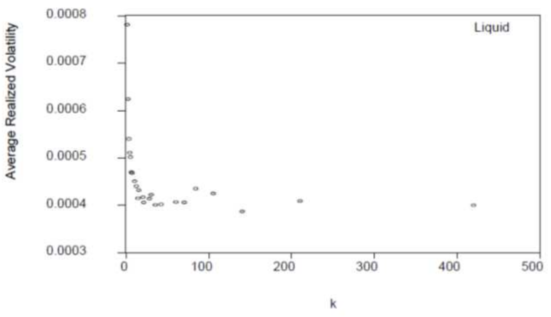
$dp(t) = \mu(t)dt + \sigma(t)dW(t)$

$\forall \xi > 0 \; \exists h > 0$:

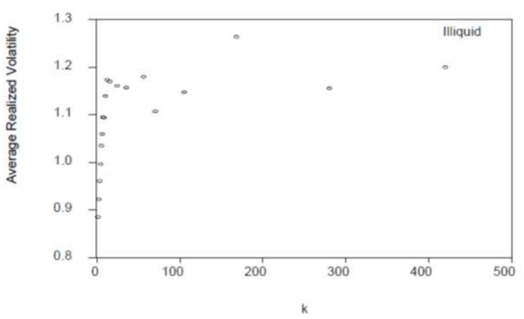
$\sup_{t-h \le \tau \le t} |\sigma^2(\tau) - \sigma^2(t)| < \xi$ (a.s.)

then $\text{plim}_{n \to \infty, h \to 0} \; \hat{\sigma}^2_{n,h,t} = \sigma^2(t)$

For liquid security, realized vol shoots up as Δ (measured in minutes k) gets small due to bid-ask bounce



For illiquid security, realized vol shoots up as Δ (measured in minutes k) gets big due to nontrading



3. Dynamic conditional correlation

Consider a collection of zero-mean GARCH(1,1) processes:

$r_{it} = \sqrt{h_{it}}\,\varepsilon_{it}$

$\varepsilon_{it} \sim$ i.i.d. $(0,1)$

$h_{it} = \omega_i + \kappa_i r_{i,t-1}^2 + \lambda_i h_{i,t-1}$

$i = 1,\ldots,n$

$q_{ijt} = s_{ij} + \alpha(\varepsilon_{i,t-1}\varepsilon_{j,t-1} - s_{ij}) + \beta(q_{ij,t-1} - s_{ij})$

$\alpha + \beta \leq 1$

If $\alpha + \beta = 1$, amounts to forecast $\varepsilon_{it}\varepsilon_{jt}$ by exponential smoothing.

$s_{ij} = E(\varepsilon_{it}\varepsilon_{jt})$ (unconditional correlation)

$\mathbf{Q}_t = (1 - \alpha - \beta)\mathbf{S} + \alpha\boldsymbol{\varepsilon}_{t-1}\boldsymbol{\varepsilon}_{t-1}' + \beta\mathbf{Q}_{t-1}$

If $\mathbf{Q}_0$ is positive definite then so is $\{\mathbf{Q}_t\}_{t=1}^T$

Define $\rho_{ijt} = \frac{q_{ijt}}{\sqrt{q_{iit}}\sqrt{q_{jjt}}}$

$$\mathbf{R}_t = \begin{bmatrix} \rho_{11t} & \cdots & \rho_{1nt} \\ \vdots & \cdots & \vdots \\ \rho_{n1t} & \cdots & \rho_{nnt} \end{bmatrix}$$

positive definite with ones along diagonal (a correlation matrix)

More generally, could consider

$$\mathbf{Q}_t = \mathbf{S} \circ (\mathbf{1}\mathbf{1}' - \mathbf{A} - \mathbf{B}) + \mathbf{A} \circ \boldsymbol{\varepsilon}_{t-1}\boldsymbol{\varepsilon}_{t-1}' + \mathbf{B} \circ \mathbf{Q}_{t-1}$$

so each correlation gets its own α_{ij}, β_{ij} instead of $\alpha_{ij} = \alpha, \beta_{ij} = \beta$.
Need $\mathbf{A}$, $\mathbf{B}$, $(\mathbf{1}\mathbf{1}' - \mathbf{A} - \mathbf{B})$ p.d.

Likelihood function for $\boldsymbol{\varepsilon}_t \sim N(\mathbf{0}, \mathbf{I}_n)$

$\Omega_t = \{\mathbf{r}_t, \mathbf{r}_{t-1}, \ldots, \mathbf{r}_1\}$

$\mathbf{r}_t | \Omega_{t-1} \sim N(\mathbf{0}, \mathbf{D}_t \mathbf{R}_t \mathbf{D}_t)$

$\mathbf{D}_t = \text{diag}\{\sqrt{h_{it}}\}$

$h_{it} = \omega_i + \kappa_i r_{i,t-1}^2 + \lambda_i h_{i,t-1}$

$\boldsymbol{\varepsilon}_t = \mathbf{D}_t^{-1} \mathbf{r}_t$

$\mathbf{Q}_t = \mathbf{S} \circ (\mathbf{1}\mathbf{1}' - \mathbf{A} - \mathbf{B}) + \mathbf{A} \circ \boldsymbol{\varepsilon}_{t-1}\boldsymbol{\varepsilon}_{t-1}' + \mathbf{B} \circ \mathbf{Q}_{t-1}$

$\mathbf{Q}_t^* = \text{diag}\{\sqrt{q_{iit}}\}$

$\mathbf{R}_t = \mathbf{Q}_t^{*-1} \mathbf{Q}_t \mathbf{Q}_t^{*-1}$

$$\mathcal{L} = -(1/2) \sum_{t=1}^{T} \{n \log(2\pi) + \log|\mathbf{D}_t \mathbf{R}_t \mathbf{D}_t| + \mathbf{r}_t' \mathbf{D}_t^{-1} \mathbf{R}_t^{-1} \mathbf{D}_t^{-1} \mathbf{r}_t\}$$

$$= -(1/2) \sum_{t=1}^{T} \{n \log(2\pi) + 2\log|\mathbf{D}_t| + \log|\mathbf{R}_t| + \boldsymbol{\varepsilon}_t' \mathbf{R}_t^{-1} \boldsymbol{\varepsilon}_t\}$$

$$= -(1/2) \sum_{t=1}^{T} \{n \log(2\pi) + 2\log|\mathbf{D}_t| + \mathbf{r}_t' \mathbf{D}_t^{-1} \mathbf{D}_t^{-1} \mathbf{r}_t - \boldsymbol{\varepsilon}_t' \boldsymbol{\varepsilon}_t + \log|\mathbf{R}_t| + \boldsymbol{\varepsilon}_t' \mathbf{R}_t^{-1} \boldsymbol{\varepsilon}_t\}$$

First component:

$-(1/2)\sum_{t=1}^{T}\{n\log(2\pi) + 2\log|\mathbf{D}_t| + \mathbf{r}_t'\mathbf{D}_t^{-1}\mathbf{D}_t^{-1}\mathbf{r}_t\}$

$= -(1/2)\sum_{i=1}^{n}\sum_{t=1}^{T}\{\log(2\pi) + \log(h_{it}) + r_{it}^2/h_{it}\}$

$h_{it} = \omega_i + \kappa_i r_{i,t-1}^2 + \lambda_i h_{i,t-1}$

can estimate $\omega_i, \kappa_i, \lambda_i$ by fitting univariate GARCH(1,1) models to series one at a time.

Second component:

$-(1/2)\sum_{t=1}^{T}\{\log|\mathbf{R}_t| + \boldsymbol{\varepsilon}_t'\mathbf{R}_t^{-1}\boldsymbol{\varepsilon}_t - \boldsymbol{\varepsilon}_t'\boldsymbol{\varepsilon}_t\}$

Can maximize with respect to correlation parameters (e.g. α, β) with $\hat{\boldsymbol{\varepsilon}}_t = \hat{\mathbf{D}}_t^{-1}\mathbf{r}_t$ for $\hat{\mathbf{D}}_t$ from first step

$\hat{\mathbf{S}} = T^{-1}\sum_{t=1}^{T}\hat{\boldsymbol{\varepsilon}}_t\hat{\boldsymbol{\varepsilon}}_t'$

$\min_{\alpha,\beta}\sum_{t=1}^{T}\{\log|\mathbf{R}_t| + \hat{\boldsymbol{\varepsilon}}_t'\mathbf{R}_t^{-1}\hat{\boldsymbol{\varepsilon}}_t\}$

$(\hat{\alpha}, \hat{\beta})$ consistent and asymptotically Normal, standard errors in Engle (2002)

VII. Time-varying variances

A. Overview
B. Extensions
C. Markov-switching GARCH

Options for Markov-switching GARCH:
(1) Gray (Journal of Financial Economics,1996)
Replace

$$h_t = \gamma_{s_t} + \alpha_{s_t} u_{t-1}^2 + \beta_{s_t} h_{t-1}$$

with

$$h_t = \gamma_{s_t} + \alpha_{s_t} u_{t-1}^2 + \beta_{s_t} \tilde{h}_{t-1}$$

$$\tilde{h}_{t-1} = \sum_{i=1}^{N} \hat{\xi}_{i,t-1|t-2} (\gamma_i + \alpha_i y_{t-2}^2 + \beta_i \tilde{h}_{t-2})$$

Options for Markov-switching GARCH:
(2) Haas, Mittnik, and Paolella
(Journal of Financial Econometrics, 2004)

$$h_{jt} = \gamma_j + \alpha_j u_{t-1}^2 + \beta_j h_{j,t-1}$$

$$y_t = \sqrt{h_{s_t,t}}\, u_t$$

Options for Markov-switching GARCH:

(3) Bauwens, Preminger, and Rombouts, (*Econometrics Journal,* 2010)-- numerical Bayesian methods

VI. Time-varying variances

A. Overview
B. Extensions
C. Markov-switching GARCH
D. Stochastic volatility

GARCH family:

$$y_t = \mathbf{x}_t' \boldsymbol{\beta} + u_t$$
$$u_t = \sqrt{h_t}\, v_t$$
$$v_t \sim \text{ i.i.d. } (0,1) \text{ (e.g. } N(0,1))$$
$$h_t = h(u_{t-1}, u_{t-2}, \ldots)$$

Implication:

the difference between the realized value y_t and its conditional expectation $\mathbf{x}_t'\boldsymbol{\beta}$ is the only information useful for forecasting the variance h_t

Stochastic volatility:

Some latent variables in addition to u_{t-j} contribute to h_t

Example:

$$y_t = \exp(h_t/2)v_t$$

$$h_t = \mu + \phi(h_{t-1} - \mu) + \sigma\eta_t$$

$$\begin{bmatrix} v_t \\ \eta_t \end{bmatrix} \sim \text{ i.i.d. } N\left(\begin{bmatrix} 0 \\ 0 \end{bmatrix}, \begin{bmatrix} 1 & 0 \\ 0 & 1 \end{bmatrix}\right)$$

argument in favor of stochastic vol:
more natural and flexible
argument in favor of GARCH:
ultimately our forecast
$E(u_t^2 | \mathbf{x}_t, y_{t-1}, \mathbf{x}_{t-1}, y_{t-2}, \ldots)$
will be some function of
$(\mathbf{x}_t, y_{t-1}, \mathbf{x}_{t-1}, y_{t-2}, \ldots)$
so why not take this function
as a primitive of the model?

Note sv model above implies

$$y_t^2 = \exp(h_t) v_t^2$$
$$\log y_t^2 = h_t + \log v_t^2$$
$$\log y_t^2 = \mu + (h_t - \mu) + \log v_t^2$$

For $\xi_t = h_t - \mu$ this is a state-space model of the form

$$\xi_t = \phi \xi_{t-1} + \sigma \eta_t$$
$$\log y_t^2 = \mu + \xi_t + \log v_t^2$$

problem: $\log v_t^2$ is not Normally distributed
solution: auxiliary particle filter

$\boldsymbol{\psi} = (\mu, \phi, \sigma)'$

$\Omega_t = \{y_t, y_{t-1}, .., y_1\}$

goal: approximate

$p(\xi_t | \Omega_t, \boldsymbol{\psi})$

$p(y_t | \Omega_{t-1}, \boldsymbol{\psi})$

Input for step $t+1$:

particles $\Lambda_t^{(i)} = \{\xi_t^{(i)}, \xi_{t-1}^{(i)}, \ldots, \xi_1^{(i)}\}$

for $i = 1, \ldots, D$ with weights $1/D$

(1) calculate measure of how useful $\xi_t^{(i)}$ is for predicting y_{t+1}

$\tilde{h}_{t+1}^{(i)} = \mu + \phi(h_t^{(i)} - \mu)$

$\tilde{\tau}_t^{(i)} = \frac{1}{\sqrt{2\pi \exp\left[\tilde{h}_{t+1}^{(i)}/2\right]}} \exp\left(\frac{-y_{t+1}^2}{2\exp\left[\tilde{h}_{t+1}^{(i)}/2\right]} \right)$

(2) Set $\tilde{\omega}_t^{(i)} = \frac{\tilde{\tau}_t^{(i)}}{\sum_{i=1}^{D} \tilde{\tau}_t(i)}$

and resample $\Lambda_t^{(j)}$ with prob $\tilde{\omega}_t^{(j)}$:

$$\Lambda_t^{(j)} = \begin{cases} \Lambda_t^{(1)} & \text{with probability } \tilde{\omega}_t^{(1)} \\ \vdots & \\ \Lambda_t^{(D)} & \text{with probability } \tilde{\omega}_t^{(D)} \end{cases}$$

(3) Generate $h_{t+1}^{(j)}$ from $N(\mu + \phi(h_t^{(j)} - \mu), \sigma^2)$ for $j = 1, \ldots, D$

(4) Calculate weights

$$\omega_{t+1}^{(j)} = \frac{1}{\tilde{\omega}_t^{(j)}} \frac{1}{\sqrt{2\pi \exp\left[h_{t+1}^{(j)}/2\right]}} \exp\left(\frac{-y_{t+1}^2}{2\exp\left[h_{t+1}^{(j)}/2\right]} \right)$$

$$\hat{p}(y_{t+1}|\Omega_t; \psi) = D^{-1} \sum_{j=1}^{D} \omega_{t+1}^{(j)}$$

$$\hat{\omega}_{t+1}^{(j)} = \frac{\omega_{t+1}^{(j)}}{D^{-1} \sum_{j=1}^{D} \omega_{t+1}^{(j)}}$$

$$\hat{E}(h_{t+1}|\Omega_{t+1}; \psi) = \sum_{j=1}^{D} \hat{\omega}_{t+1}^{(j)} h_{t+1}^{(j)}$$

(5) Resample

$$\Lambda_{t+1}^{(i)} = \begin{cases} \Lambda_{t+1}^{(1)} & \text{with probability } \hat{\omega}_{t+1}^{(1)} \\ \vdots & \\ \Lambda_{t+1}^{(D)} & \text{with probability } \hat{\omega}_{t+1}^{(D)} \end{cases}$$

$$\mathcal{L}(\boldsymbol{\psi}) = \sum_{t=0}^{T-1} \log \hat{p}(y_{t+1}|\Omega_t;\boldsymbol{\psi})$$

Note structure is no more difficult for generalizations, e.g.,

$y_t = \mathbf{x}_t'\boldsymbol{\beta} + \exp(h_t/2)v_t$

$v_t \sim$ Student t $(0, 1, \eta)$

Just replace $N(0, \exp(h_t/2))$ densities above with Student t $(\mathbf{x}_t'\boldsymbol{\beta}, \exp(h_t/2), \eta)$

Alternatively, other tricks could allow us to use linear state-space methods as building blocks

Return to original problem:

$$\xi_t = \phi \xi_{t-1} + \sigma \eta_t$$

$$\log y_t^2 = \mu + \xi_t + \log v_t^2$$

$$v_t \sim N(0,1)$$

$$\log v_t^2 = \log \chi^2(1)$$

$$z_t = \log v_t^2 = \log \chi^2(1)$$

can approximate this density arbitrarily well with a mixture of Normals

$p(z_t) = \sum_{i=1}^{K} \frac{\pi_i}{\sqrt{2\pi}\,\tau_i} \exp\left[-\frac{(z_t - \delta_i)}{2\tau_i^2}\right]$

$K = 7$ gives excellent approximation

values of π_i, τ_i, δ_i are numerically known (not a function of data, instead function of $\log[\chi^2(1)]$ distribution)

We could generate a value for z_t from this distribution in two steps

step 1: generate $s_t \in \{1, 2, \ldots, K\}$

Prob$(s_t = i) = \pi_i$

step 2: generate $z_t | s_t \sim N(\delta_{s_t}, \tau_{s_t}^2)$

$\xi_t = \phi \xi_{t-1} + \sigma \eta_t$

$\log y_t^2 = \mu + \xi_t + z_t$

$z_t | s_t \sim N(\delta_{s_t}, \tau_{s_t}^2)$

Conditional on $\mathbf{s} = (s_1, .., s_T)'$ this is Gaussian linear state-space model

$\boldsymbol{\psi} = (\mu, \phi, \sigma)'$

Sampling from $p(\mathbf{y}|\boldsymbol{\xi}, \mathbf{s}, \boldsymbol{\psi})$ or $p(\boldsymbol{\xi}|\mathbf{y}, \mathbf{s}, \boldsymbol{\psi})$ are standard

$$p(\mathbf{s}|\mathbf{y}, \boldsymbol{\xi}, \boldsymbol{\psi}) = \prod_{t=1}^{T} p(s_t|z_t)$$

$$p(s_t = j|z_t) = \frac{(\pi_j/\tau_j)\exp\left[-\frac{(z_t-\delta_j)^2}{2\tau_j^2}\right]}{\sum_{i=1}^{K}(\pi_i/\tau_i)\exp\left[-\frac{(z_t-\delta_i)^2}{2\tau_i^2}\right]}$$

(recall π_j, δ_j, τ_j are all known properties of $\log\chi^2(1)$ distribution)

Summary of linear Gaussian representation:

$$\xi_t = \phi\xi_{t-1} + \sigma\eta_t$$

$$y_t^* = \mu + \xi_t + z_t$$

$$z_t \sim N(0, \tau_{s_t}^2)$$

$$y_t^* = \log y_t^2 - \delta_{s_t}$$