V. Nonlinear state-space models

A. Extended Kalman filter
B. Particle filter
C. Solution and estimation of nonlinear dynamic stochastic general equilibrium models
   1. Motivation

$\mathbf{x}_t$ = vector of exogenous variables

$\boldsymbol{\varepsilon}_t$ = vector of exogenous disturbances

$\mathbf{f}(\mathbf{x}_{t+1}, \mathbf{x}_t, \boldsymbol{\varepsilon}_{t+1}) = \mathbf{0}$

(equation of motion for $\mathbf{x}_t$)

$\mathbf{z}_t$ = vector of endogenous variables

$E_t \mathbf{a}(\mathbf{z}_{t+1}, \mathbf{z}_t, \mathbf{x}_t) = \mathbf{0}$

(equations derived from econ theory)

Approach we discussed earlier:

(1) Log-linearize system.

$$\mathbf{A} E_t \mathbf{z}_{t+1} = \mathbf{B}\mathbf{z}_t + \mathbf{C}\mathbf{x}_t$$

$$\mathbf{x}_{t+1} = \boldsymbol{\Phi}\mathbf{x}_t + \boldsymbol{\varepsilon}_{t+1}$$

(2) Find rational-expectations solution.
predetermined component: $\mathbf{z}_{1t}$

$$\mathbf{z}_{1,t+1} = \mathbf{H}_{11}\mathbf{z}_{1t} + \mathbf{H}_{12}\mathbf{x}_t$$

forward-looking component: $\mathbf{z}_{2t}$

$$\mathbf{z}_{2t} = \mathbf{H}_{21}\mathbf{z}_{1t} + \mathbf{H}_{22}\mathbf{x}_t$$

(3) Recognize as state-space system.

$\mathbf{y}_t$ = observed elements of $\{\mathbf{z}_t, \mathbf{x}_t\}$

$\boldsymbol{\xi}_t$ = unobserved elements of $\{\mathbf{z}_t, \mathbf{x}_t\}$

$$\boldsymbol{\xi}_{t+1} = \boldsymbol{\Phi}\boldsymbol{\xi}_t + \mathbf{v}_{t+1}$$

$$\mathbf{y}_t = \mathbf{a} + \mathbf{H}'\boldsymbol{\xi}_t + \mathbf{w}_t$$

(4) Estimate parameters by MLE
or Bayesian methods.

Things we lose from linearization:
(1) Statistical representation of recessions.
Recall that a discrete Markov chain can
be viewed as VAR(1).

Things we lose from linearization:
(2) Economic characterization of risk aversion.

$$1 = E_t\left[\frac{\beta U'(c_{t+1})(1+r_{j,t+1})}{U'(c_t)}\right]$$

for $r_{j,t+1}$ the real return on any asset.
Finance: different assets have different expected returns due to covariance between $r_{j,t+1}$ and c_{t+1}

$$1 = E_t\left[\frac{\beta U'(c_{t+1})(1+r_{j,t+1})}{U'(c_t)}\right]$$

steady state:

$$1 = \frac{\beta U'(c)(1+r_j)}{U'(c)}$$

$\beta(1+r_j) = 1$ for all j

linearization around steady state

$$U'(c_t) = E_t[\beta U'(c_{t+1})(1+r_{j,t+1})]$$
$$\simeq (1+r)\beta U''(c)E_t(c_{t+1}-c)$$
$$+\beta U'(c)E_t(r_{j,t+1}-r)$$

same for all j

Things we lose from linearization:

(3) Role of changes in uncertainty, time-varying volatility.

(4) Behavior of economy when interest rate is at zero lower bound

$R_t = \min(R_t^*, \bar{R})$

Approaches to estimating nonlinear dynamic economic models.

Step 1: Find approximating nonlinear state-space representation using either

(1) perturbation methods (e.g., Fernandez-Villaverde and Rubio-Ramirez), or

(2) projection methods (e.g., Gust, Lopez-Salido, and Smith)

Step 2: Estimate parameters using particle filter or other nonlinear estimation (MLE or Bayesian)

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Example:

$$\max_{\{c_t,k_{t+1}\}_{t=0}^{\infty}} E_0 \sum_{t=0}^{\infty} \beta^t \log c_t$$

s.t. $c_t + k_{t+1} = e^{z_t} k_t^{\alpha} + (1-\delta)k_t \quad t = 1,2,\ldots$

$z_t = \rho z_{t-1} + \sigma \varepsilon_t \quad t = 1,2,\ldots$

k_0, z_0 given

$\varepsilon_t \sim N(0,1)$

Approach: we will consider a continuum of economies indexed by σ and study solutions as $\sigma \to 0$ (that is, as economy becomes deterministic).

We seek decision rules of the form

$c_t = c(k_t, z_t; \sigma)$

$k_{t+1} = k(k_t, z_t; \sigma)$

Write F.O.C. as $E_t \mathbf{a}(k_t, z_t; \sigma, \varepsilon_{t+1}) = \mathbf{0}$

$$a_1(k_t, z_t; \sigma, \varepsilon_{t+1}) = \frac{1}{c(k_t,z_t;\sigma)} - \beta \frac{\alpha k(k_t,z_t;\sigma)^{\alpha-1} \exp(\rho z_t + \sigma\varepsilon_{t+1})}{c(k(k_t,z_t;\sigma), \rho z_t + \sigma\varepsilon_{t+1}; \sigma)}$$

$$a_2(k_t, z_t; \sigma, \varepsilon_{t+1}) = c(k_t, z_t; \sigma) + k(k_t, z_t; \sigma) - e^{z_t} k_t^{\alpha} - (1-\delta)k_t$$

Zero-order approximation
(deterministic steady state)

$\sigma = 0$

$z_t = z = 0$

$k_t = k$

$\mathbf{a}(k, 0; 0) = \mathbf{0}$

$a_1(k, 0; 0) = 0$

$\Rightarrow \frac{1}{c} - \beta \frac{\alpha k^{\alpha-1}}{c} = 0$

$\Rightarrow 1 = \beta \alpha k^{\alpha-1}$

$a_2(k, 0; 0) = 0$

$\Rightarrow c + k - k^{\alpha} - (1-\delta)k$

$\Rightarrow c = k^{\alpha} - \delta k$

First-order approximation:
Since $E_t \mathbf{a}(k_t, z_t; \sigma, \varepsilon_{t+1}) = \mathbf{0}$ for all $k_t, z_t; \sigma$, it follows that

$E_t \mathbf{a}_k(k_t, z_t; \sigma, \varepsilon_{t+1}) = \mathbf{0}$

for $\mathbf{a}_k(k_t, z_t; \sigma, \varepsilon_{t+1}) = \frac{\partial \mathbf{a}(k_t, z_t; \sigma, \varepsilon_{t+1})}{\partial k_t}$

likewise

$E_t \mathbf{a}_z(k_t, z_t; \sigma, \varepsilon_{t+1}) = E_t \mathbf{a}_\sigma(k_t, z_t; \sigma, \varepsilon_{t+1}) = \mathbf{0}$

$$E_t\left\{\frac{\partial a_1(k_t,z_t;\sigma,\varepsilon_{t+1})}{\partial k_t}\Big|_{k_t=k,z_t=0,\sigma=0}\right\} =$$

$$\frac{-1}{c^2}c_k - \frac{\beta\alpha(\alpha-1)k^{\alpha-2}}{c}k_k + \frac{\beta\alpha k^{\alpha-1}}{c^2}c_k k_k$$

Since c and k are known from previous step, setting this to zero gives us an equation in the unknowns c_k and k_k where for example

$$c_k = \frac{\partial c(k_t,z_t;\sigma)}{\partial k_t}\Big|_{k_t=k,z_t=0,\sigma=0}$$

$$\frac{\partial a_2(k_t,z_t;\sigma)}{\partial k_t}\Big|_{k_t=k,z_t=0,\sigma=0} =$$

$$c_k + k_k - \alpha k^{\alpha-1} - (1-\delta)$$

This is a second equation in $c_k, k_k,$ which together with the first can now be solved for c_k, k_k as a function of c and k

$$E_t\left\{\frac{\partial a_1(k_t,z_t;\sigma,\varepsilon_{t+1})}{\partial z_t}\Big|_{k_t=k,z_t=0,\sigma=0}\right\} =$$

$$\frac{-1}{c^2}c_z - \frac{\beta\alpha(\alpha-1)k^{\alpha-2}}{c}k_z - \frac{\beta\alpha k^{\alpha-1}\rho}{c}$$

$$+ \frac{\beta\alpha k^{\alpha-1}}{c^2}(c_k k_z + \rho c_z)$$

$$\frac{\partial a_2(k_t,z_t;\sigma)}{\partial z_t}\Big|_{k_t=k,z_t=0,\sigma=0} =$$

$$c_z + k_z - k^{\alpha}$$

setting these to zero allows us to solve for c_z, k_z

$$\frac{\partial a_1(k_t,z_t;\sigma,\varepsilon_{t+1})}{\partial\sigma}\Big|_{k_t=k,z_t=0,\sigma=0} =$$

$$\frac{-1}{c^2}c_\sigma - \frac{\beta\alpha(\alpha-1)k^{\alpha-2}}{c}k_\sigma - \frac{\beta\alpha k^{\alpha-1}\varepsilon_{t+1}}{c}$$

$$+ \frac{\beta\alpha k^{\alpha-1}}{c^2}(c_k k_\sigma + \varepsilon_{t+1}c_z + c_\sigma)$$

$$\frac{\partial a_2(k_t,z_t;\sigma)}{\partial\sigma}\Big|_{k_t=k,z_t=0,\sigma=0} =$$

$$c_\sigma + k_\sigma$$

Taking expectations and setting to zero yields

$$\frac{-1}{c^2}c_\sigma - \frac{\beta\alpha(\alpha-1)k^{\alpha-2}}{c}k_\sigma$$

$$+ \frac{\beta\alpha k^{\alpha-1}}{c^2}(c_k k_\sigma + c_\sigma) = 0$$

$$c_\sigma + k_\sigma = 0$$

which has solution $c_\sigma = k_\sigma = 0$

$\Rightarrow$ volatility, risk aversion play no role in first-order approximation

Now that we've calculated derivatives, we have the approximate solutions

$c(k_t,z_t;\sigma) \simeq c + c_k(k_t - k) + c_z z_t + c_\sigma\sigma$

$k(k_t,z_t;\sigma) \simeq k + k_k(k_t - k) + k_z z_t + k_\sigma\sigma$

where we showed that $c_\sigma = k_\sigma = 0$

Thus, first-order perturbation is a way to find linearization or log-linearization

But we don't have to stop here. Since $E_t\mathbf{a}(k_t, z_t; \sigma, \varepsilon_{t+1}) = \mathbf{0}$ for all k_t, z_t, σ, second derivatives with respect to $(k_t, z_t; \sigma)$ also have to be zero.

Differentiate each of the 6 equations

$E_t\mathbf{a}_k(k_t, z_t; \sigma, \varepsilon_{t+1}) = \mathbf{0}$

$E_t\mathbf{a}_z(k_t, z_t; \sigma, \varepsilon_{t+1}) = \mathbf{0}$

$E_t\mathbf{a}_\sigma(k_t, z_t; \sigma, \varepsilon_{t+1}) = \mathbf{0}$

with respect to $k_t, z_t,$ and σ.

Gives 18 linear equations in the 12 unknowns $\{c_{ij}, k_{ij}\}_{i,j \in \{k,z,\sigma)}$ with 6 equations redundant by symmetry of second derivatives (e.g., $c_{kz} = c_{zk}$) and where coefficients on c_{ij}, k_{ij} are known from previous step

We then have second-order approximation to decision functions,

$$c(k_t, z_t; \sigma) \simeq c + \mathbf{c}_1' \mathbf{s}_t + (1/2)\mathbf{s}_t' \mathbf{C}_2 \mathbf{s}_t$$

$$k(k_t, z_t; \sigma) \simeq k + \mathbf{k}_1' \mathbf{s}_t + (1/2)\mathbf{s}_t' \mathbf{K}_2 \mathbf{s}_t$$

$$\mathbf{c}_1' = \begin{bmatrix} c_k & c_z & 0 \end{bmatrix}$$

$$\mathbf{s}_t = \begin{bmatrix} (k_t - k) & z_t & \sigma \end{bmatrix}$$

$$\mathbf{k}_1' = \begin{bmatrix} k_k & k_z & 0 \end{bmatrix}$$

$$\mathbf{C}_2 = \begin{bmatrix} c_{kk} & c_{kz} & c_{k\sigma} \\ c_{zk} & c_{zz} & c_{z\sigma} \\ c_{\sigma k} & c_{\sigma z} & c_{\sigma\sigma} \end{bmatrix}$$

$$\mathbf{K}_2 = \begin{bmatrix} k_{kk} & k_{kz} & k_{k\sigma} \\ k_{zk} & k_{zz} & k_{z\sigma} \\ k_{\sigma k} & k_{\sigma z} & k_{\sigma\sigma} \end{bmatrix}$$

$$c(k_t, z_t; \sigma) \simeq c + \mathbf{c}_1' \mathbf{s}_t + (1/2)\mathbf{s}_t' \mathbf{C}_2 \mathbf{s}_t$$

$$\mathbf{s}_t = \begin{bmatrix} (k_t - k) & z_t & \sigma \end{bmatrix}$$

Note: term on σ^2 in $\mathbf{s}_t' \mathbf{C}_2 \mathbf{s}_t$ acts like another constant reflecting precautionary behavior left out of certainty-equivalence steady-state c

We could in principle continue to as high an order approximation as we wanted

V. Nonlinear state-space models

D. Nonlinear DSGE's

1. Motivation
2. Perturbation methods
3. Illustration

Using a particle filter to estimate a DSGE with second-order perturbation approximation (Fernandez-Villaverde and Rubio-Ramirez, REStud, 2007)

$$
\begin{aligned}
C_t + I_t &= A_t K_t^{\alpha} L_t^{1-\alpha} \\
K_{t+1} &= (1-\delta) K_t + U_t I_t \\
\log A_t &= \zeta + \log A_{t-1} + \sigma_{at}\varepsilon_{at} \\
\log U_t &= \theta + \log U_{t-1} + \sigma_{vt}\varepsilon_{vt} \\
\log \sigma_{at} &= (1-\lambda_a)\log \overline{\sigma}_a \\
&\quad + \lambda_a \log \sigma_{a,t-1} + \tau_a \eta_{at} \\
\log \sigma_{vt} &= (1-\lambda_v)\log \overline{\sigma}_v \\
&\quad + \lambda_v \log \sigma_{v,t-1} + \tau_v \eta_{vt}
\end{aligned}
$$

$$E_0 \sum_{t=0}^{\infty} \beta^t \{e^{d_t} \log C_t + \psi \log(1 - L_t)\}$$
$$d_t = \rho d_{t-1} + \sigma_{dt}\varepsilon_{dt}$$
$$\log \sigma_{dt} = (1 - \lambda_d) \log \overline{\sigma}_d + \lambda_d \log \sigma_{d,t-1} + \tau_d \eta_{dt}$$

$$\mathbf{v}_t = (\varepsilon_{at}, \varepsilon_{vt}, \varepsilon_{dt}, \eta_{at}, \eta_{vt}, \eta_{dt})'$$
$$\mathbf{v}_t \sim N(\mathbf{0}, \mathbf{I}_6)$$
$\mathbf{\Omega} = \text{diag}\{\overline{\sigma}_a^2, \overline{\sigma}_v^2, \overline{\sigma}_d^2, \tau_a^2, \tau_v^2, \tau_d^2\}$

perturbation method: Continuum of economies with variance $\chi\mathbf{\Omega}$, take expansion around $\chi = 0$

Transformations to find steady-state representation:

$$Z_t = A_{t-1}^{1/(1-\alpha)} U_{t-1}^{\alpha/(1-\alpha)}$$
$$\tilde{Y}_t = Y_t/Z_t,\ \tilde{C}_t = C_t/Z_t,\ \tilde{I}_t = I_t/Z_t$$
$$\tilde{U}_t = U_t/U_{t-1},\ \tilde{A}_t = A_t/A_{t-1},\ \tilde{K}_t = K_t/Z_t U_{t-1}$$
$\tilde{k}$ = log of steady-state value for $\tilde{K}$
$$\hat{\tilde{k}}_t = \log \tilde{K}_t - \tilde{k}$$

state vector for economic model:

$$\tilde{\mathbf{s}}_t = (\hat{\tilde{k}}_t, \varepsilon_{at}, \varepsilon_{vt}, \varepsilon_{dt}, d_{t-1},$$
$$\sigma_{at} - \overline{\sigma}_a, \sigma_{vt} - \overline{\sigma}_v, \sigma_{dt} - \overline{\sigma}_d)'$$

second-order perturbation:

$$\hat{\tilde{k}}_{t+1} = \boldsymbol{\psi}_{k1}'\tilde{\mathbf{s}}_t + (1/2)\tilde{\mathbf{s}}_t'\boldsymbol{\Psi}_{k2}\tilde{\mathbf{s}}_t + \psi_{k0}$$
$$\hat{i}_t = \boldsymbol{\psi}_{i1}'\tilde{\mathbf{s}}_t + (1/2)\tilde{\mathbf{s}}_t'\boldsymbol{\Psi}_{i2}\tilde{\mathbf{s}}_t + \psi_{i0}$$
$$\hat{\tilde{\ell}}_t = \boldsymbol{\psi}_{\ell 1}'\tilde{\mathbf{s}}_t + (1/2)\tilde{\mathbf{s}}_t'\boldsymbol{\Psi}_{\ell 2}\tilde{\mathbf{s}}_t + \psi_{\ell 0}$$

ψ_{j0} reflects precautionary effects

However, we will observe actual GDP growth per capita

$$\Delta \log Y_t = \Delta \log \tilde{Y}_t + \tfrac{1}{1-\alpha}(\Delta \log A_{t-1} + \alpha \Delta \log U_{t-1}) + \sigma_{y\varepsilon}\varepsilon_{yt}$$
$$= h_y(\tilde{\mathbf{s}}_t, \tilde{\mathbf{s}}_{t-1}) + \sigma_{y\varepsilon}\varepsilon_{yt}$$

ε_{yt} = measurement error

Also observe real gross investment per capita (I_t), hours worked per capita (ℓ_t), and relative price of investment goods P_t

$$\Delta \log I_t = h_i(\tilde{\mathbf{s}}_t, \tilde{\mathbf{s}}_{t-1}) + \sigma_{i\varepsilon}\varepsilon_{it}$$
$$\log \ell_t = h_\ell(\tilde{\mathbf{s}}_t, \tilde{\mathbf{s}}_{t-1}) + \sigma_{\ell\varepsilon}\varepsilon_{\ell t}$$
$$\Delta \log P_t = -\Delta \log U_t$$

$$\tilde{\mathbf{s}}_t = (\hat{\tilde{k}}_t, \varepsilon_{at}, \varepsilon_{vt}, \varepsilon_{dt}, d_{t-1},$$
$$\sigma_{at} - \overline{\sigma}_a, \sigma_{vt} - \overline{\sigma}_v, \sigma_{dt} - \overline{\sigma}_d)'$$
$$\mathbf{v}_t = (\varepsilon_{at}, \varepsilon_{vt}, \varepsilon_{dt}, \eta_{at}, \eta_{vt}, \eta_{dt})'$$
$$\mathbf{S}_t = (\tilde{\mathbf{s}}_t', \tilde{\mathbf{s}}_{t-1}')$$

state equation

$$\mathbf{S}_t = \mathbf{f}(\mathbf{S}_{t-1}, \mathbf{v}_t)$$
$$f_1(\mathbf{S}_{t-1}, \mathbf{v}_t) = \boldsymbol{\psi}_{k1}'\tilde{\mathbf{s}}_t + (1/2)\tilde{\mathbf{s}}_t'\boldsymbol{\Psi}_{k2}\tilde{\mathbf{s}}_t + \psi_{k0}$$

$$f_2(\mathbf{S}_{t-1}, \mathbf{v}_t) = \varepsilon_{at}$$
$$\vdots$$
$$f_5(\mathbf{S}_{t-1}, \mathbf{v}_t) = \rho d_{t-2} + \sigma_{d,t-1}\varepsilon_{d,t-1}$$
$$f_6(\mathbf{S}_{t-1}, \mathbf{v}_t) = \exp[(1-\lambda_a)\log\overline{\sigma}_a$$
$$+ \lambda_a \log \sigma_{a,t-1} + \tau_a \eta_{at}] - \overline{\sigma}_a$$
$$\vdots$$
$$\mathbf{f}_{9-16}(\mathbf{S}_{t-1}, \mathbf{v}_t) = \tilde{\mathbf{s}}_{t-1}$$

$$\mathbf{y}_t = (\Delta \log Y_t, \Delta \log I_t, \log \ell_t, \Delta \log P_t)'$$

observation equation:

$$\mathbf{y}_t = \mathbf{h}(\mathbf{S}_t) + \mathbf{w}_t$$

According to the set-up, ε_{vt} is observed directly from the change in investment price each period

$$\log U_t = \theta + \log U_{t-1} + \sigma_{vt}\varepsilon_{vt}$$

$$\Delta \log P_t = -\Delta \log U_t$$

We only need to generate a draw for

$$\mathbf{v}_{1t} = (\varepsilon_{at}, \varepsilon_{dt}, \eta_{at}, \eta_{vt}, \eta_{dt})'$$

in order to have a value for σ_{vt} and value for ε_{vt}

$$\varepsilon_{vt} = -\frac{\Delta \log P_t + \theta}{\sigma_{vt}}$$

Initialization:

$$\mathbf{S}_t = \mathbf{f}(\mathbf{S}_{t-1}, \mathbf{v}_t)$$

One approach is to set $\mathbf{S}_{-N} = \mathbf{0}$, draw $\mathbf{v}_{-N+1}, \mathbf{v}_{-N+2}, \ldots, \mathbf{v}_0$ from $N(\mathbf{0}, \mathbf{I}_6)$ to obtain D draws (particles) for $\{\mathbf{S}_0^{(i)}\}_{i=1}^{D}$

Estimation using bootstrap particle filter

As of date t we have calculated a set

$\Lambda_t^{(i)} = \{\mathbf{S}_t^{(i)}, \mathbf{S}_{t-1}^{(i)}, \ldots, \mathbf{S}_0^{(i)}\})$

for $i = 1, \ldots, D$

To update for $t+1$ we do the following:

Step 1: generate $\mathbf{v}_{1,t+1}^{(i)} \sim N(\mathbf{0}, \mathbf{I}_5)$ for $i = 1, \ldots, D$

Step 2: generate $\mathbf{S}_{t+1}^{(i)} = \mathbf{f}\left(\mathbf{S}_t^{(i)}, \mathbf{v}_{t+1}^{(i)}\right)$ except for the third element $\varepsilon_{v,t+1}^{(i)}$

Step 3: calculate

$\mathbf{w}_{t+1}^{(i)} = \mathbf{y}_{t+1} - \mathbf{h}(\mathbf{S}_{t+1}^{(i)})$

and set third element of $\mathbf{S}_{t+1}^{(i)}$ equal to fourth element of $\mathbf{w}_{t+1}^{(i)}$, $\varepsilon_{v,t+1}^{(i)} = -\frac{\Delta \log P_{t+1} + \theta}{\sigma_{v,t+1}^{(i)}}$

Step 4: calculate

$$\tilde{\omega}_{t+1}^{(i)} = (2\pi)^{-4/2} |\mathbf{D}_{t+1}^{(i)}| \times \exp\left(-(1/2)\left[\mathbf{w}_{t+1}^{(i)}\right][\mathbf{D}_{t+1}^{(i)}]^{-1}\left[\mathbf{w}_{t+1}^{(i)}\right]\right)$$

$$\mathbf{D}_{t+1}^{(i)} = \begin{bmatrix} \sigma_{y\varepsilon}^2 & 0 & 0 & 0 \\ 0 & \sigma_{i\varepsilon}^2 & 0 & 0 \\ 0 & 0 & \sigma_{\ell\varepsilon}^2 & 0 \\ 0 & 0 & 0 & \left[\sigma_{v,t+1}^{(i)}\right]^2 \end{bmatrix}$$

Step 5: Contribution to likelihood is

$$\hat{p}(\mathbf{y}_{t+1}|\Omega_t) = D^{-1}\sum_{i=1}^{D}\tilde{\omega}_{t+1}^{(i)} = \overline{\omega}_{t+1}$$

Step 6: Calculate $\hat{\omega}_{t+1}^{(i)} = \tilde{\omega}_{t+1}^{(i)}/\overline{\omega}_{t+1}$ and resample

$$\Lambda_{t+1}^{(j)} = \begin{cases} \Lambda_{t+1}^{(1)} & \text{with probability } \hat{\omega}_{t+1}^{(1)} \\ \vdots & \\ \Lambda_{t+1}^{(D)} & \text{with probability } \hat{\omega}_{t+1}^{(D)} \end{cases}$$

Structural parameters:

$$\boldsymbol{\theta} = (\alpha,\delta,\rho,\beta,\psi,\theta,\zeta,\tau_a,\tau_v,\tau_d, \overline{\sigma}_a,\overline{\sigma}_v,\overline{\sigma}_d,\lambda_a,\lambda_v,\lambda_d,\sigma_{y\varepsilon},\sigma_{i\varepsilon},\sigma_{\ell\varepsilon})'$$

Fernandez-Villaverde and Rubio-Ramirez estimate $\boldsymbol{\theta}$ by maximizing

$$\hat{\mathcal{L}}(\boldsymbol{\theta}) = \sum_{t=1}^{T}\hat{p}(\mathbf{y}_t|\Omega_{t-1};\boldsymbol{\theta})$$