III. Linear state-space models

A. State-space representation of a dynamic system
B. Kalman filter
C. Using the Kalman filter
D. Bayesian analysis of linear state-space models
E. Solutions to linear rational expectations models
F. Estimating DSGE models

instantaneous utility function:

$$U_t = a_t^b \Big[\tfrac{1}{1-\lambda_c} (C_t - hC_{t-1})^{1-\lambda_c} - \tfrac{a_t^L}{1+\lambda_\ell} (\ell_t)^{1+\lambda_\ell} \Big]$$

$h > 0 \quad \Rightarrow$ habit persistence

$a_t^b = \rho_b a_{t-1}^b + \eta_t^b \qquad \eta_t^b \sim$ i.i.d. $N(0, \sigma_b^2)$

$\Rightarrow$ shock to intertemporal subs

$a_t^L = \rho_L a_{t-1}^L + \eta_t^L$

$\Rightarrow$ shock to intratemporal subs

Let $\widehat{C}_t$ denote deviation of $\log(C_t)$ from its steady-state value

(1) $$\widehat{C}_t = \left(\tfrac{h}{1+h}\right)\widehat{C}_{t-1} + \left(\tfrac{1}{1+h}\right)E_t\widehat{C}_{t+1} - \tfrac{(1-h)}{(1+h)\lambda_c}\left(\hat{R}_t - E_t\hat{\pi}_{t+1}\right) + \tfrac{(1-h)}{(1+h)\lambda_c}(\hat{a}_t^b - E_t\hat{a}_{t+1}^b)$$

capital evolution:

$K_t = K_{t-1}(1-\delta) + [1 - S(a_t^I I_t / I_{t-1})] I_t$

$S(.)$ = adjustment costs

$a_t^I = \rho_I a_{t-1}^I + \eta_t^I$

(2) $\hat{K}_t = (1-\delta)\hat{K}_{t-1} + \delta \hat{I}_{t-1}$

(3) $\hat{I}_t = \left(\frac{1}{1+\beta}\right)\hat{I}_{t-1} + \left(\frac{\beta}{1+\beta}\right)E_t\hat{I}_{t+1}$

$+\frac{\varphi}{1+\beta}\hat{Q}_t - \frac{\beta E_t \hat{a}_{t+1}^I - \hat{a}_t^I}{1+\beta}$

$\varphi = 1/S''$

Q_t = value of capital stock

(4) $\hat{Q}_t = -\left(\hat{R}_t - \hat{\pi}_{t+1}\right) + \frac{1-\delta}{1-\delta+\overline{r}_k} E_t \hat{Q}_{t+1}$

$+\frac{\overline{r}_k}{1-\delta+\overline{r}_k} E_t \hat{r}_{K,t+1} + \eta_t^Q$

r_{Kt} = rate of return to capital

η_t^Q = tacked on

output from producer of intermediate good of type j

$y_t^j = a_t^a K_{t-1}(j)^\alpha L_t(j)^{1-\alpha} - \Phi$

Φ = fixed cost

a_t^a = productivity shock

$a_t^a = \rho_a a_{t-1}^a + \eta_t^a$

$L_t(j)$ = aggregate of labor hired from each household τ

$L_t(j) = \left\{\int_0^1 [\ell_t(\tau)]^{1/(1+\lambda_{w,t})} d\tau\right\}^{1+\lambda_{w,t}}$

$\lambda_{w,t} = \lambda_w + \eta_t^w$

η_t^w = shock to workers' market power

wage stickiness:

a fraction ξ_w of workers are not allowed to change their wage but instead have their wage increase from the previous value by $(P_{t-1}/P_{t-2})^{\gamma_w}$

γ_w = degree of indexing

(5) $\widehat{w}_t = \frac{\beta}{1+\beta} E\widehat{w}_{t+1} + \frac{1}{1+\beta}\widehat{w}_{t-1}$
$+\frac{\beta}{1+\beta} E_t\hat{\pi}_{t+1} - \frac{1+\beta\gamma_w}{1+\beta}\hat{\pi}_t + \frac{\gamma_w}{1+\beta}\hat{\pi}_{t-1}$
$-h_w\Big[\widehat{w}_t - \lambda_L\hat{L}_t - \frac{\lambda_c}{1-h}(\widehat{C}_t - h\widehat{C}_{t-1})$
$-\hat{a}_t^L - \hat{\eta}_t^w]$
$h_w = \frac{1}{1+\beta}\frac{(1-\beta\xi_w)(1-\xi_w)}{[1+(1+\lambda_w)\lambda_L/\lambda_w]\xi_w}$

labor demand
$\frac{W_t L_t(j)}{r_{K,t} z_t K_{t-1}(j)} = \frac{1-\alpha}{\alpha}$
z_t = capital utilization
(6) $\hat{L}_t = -\widehat{w}_t + (1+\psi)\hat{r}_{K,t} + \hat{K}_{t-1}$
ψ = parameter based on cost of utilizing capital

intermediate goods sold to final goods producer with market power of firm j governed by
$\lambda_{p,t} = \lambda_p + \eta_t^p$
ξ_p = fraction allowed to adjust prices
γ_p = indexing parameter

(7) $\hat{\pi}_t = \frac{\beta}{1+\beta\gamma_p} E_t \hat{\pi}_{t+1} + \frac{\gamma_p}{1+\beta\gamma_p} \hat{\pi}_{t-1}$

$+h_p \left[\alpha \hat{r}_{K,t} + (1-\alpha)\widehat{w}_t - \hat{a}_t^a + \eta_t^p \right]$

$h_p = \frac{1}{1+\beta\gamma_p} \frac{(1-\beta\xi_p)(1-\xi_p)}{\xi_p}$

goods market equilibrium

(8) $\widehat{Y}_t = [1 - \delta(\overline{K}/\overline{Y}) - (\overline{G}/\overline{Y})]\widehat{C}_t$

$+\delta(\overline{K}/\overline{Y})\hat{I}_t + (\overline{G}/\overline{Y})\hat{a}_t^G$

production function then determines $r_{K,t}$

(9) $\widehat{Y}_t = \phi \hat{a}_t^a + \phi\alpha \hat{K}_{t-1} + \phi\alpha\psi \hat{r}_{K,t}$

$+\phi(1-\alpha)\hat{L}_t$

$\phi = 1 + \frac{\Phi}{\text{s.s. costs}}$

monetary policy (Taylor Rule)

(10) $\hat{R}_t = \rho \hat{R}_{t-1} + (1-\rho)\{\overline{\pi}_t +$

$r_\pi(\hat{\pi}_{t-1} - \overline{\pi}_{t-1}) + r_Y(\widehat{Y}_t - \widehat{Y}_t^P)\}$

$+r_{\Delta\pi}(\hat{\pi}_t - \hat{\pi}_{t-1})$

$+ \ r_{\Delta Y}\left[\widehat{Y}_t - \widehat{Y}_t^p - (\widehat{Y}_{t-1} - \widehat{Y}_{t-1}^p) \right] + \eta_t^R$

$\overline{\pi}_t =$ inflation target

$\overline{\pi}_t = \rho_\pi \overline{\pi}_{t-1} + \eta_t^\pi$

$\widehat{Y}_t^p$ = output level if prices perfectly flexible

$\mathbf{y}_t = (\widehat{C}_t, \widehat{C}_{t-1}, \hat{R}_t, \hat{R}_{t-1}, \hat{K}_t, \hat{K}_{t-1}, \hat{I}_t, \hat{I}_{t-1},$
$\hat{Q}_t, \widehat{w}_t, \widehat{w}_{t-1}, \hat{L}_t, \hat{\pi}_t, \hat{\pi}_{t-1}, \widehat{Y}_t, \hat{r}_{K,t})'$

$\mathbf{x}_t = (\hat{a}_t^b, \hat{a}_t^I, \eta_t^Q, \hat{a}_t^L, \eta_t^w, \hat{a}_t^a, \eta_t^p, \hat{a}_t^G, \overline{\pi}_t, \eta_t^R)'$

equations (1)-(10) (along with lag definitions) can be written as

$\mathbf{A}E_t\mathbf{y}_{t+1} = \mathbf{B}\mathbf{y}_t + \mathbf{C}\mathbf{x}_t$

while shocks satisfy

$\mathbf{x}_{t+1} = \mathbf{\Phi}\mathbf{x}_t + \boldsymbol{\varepsilon}_{t+1}$

(note also $E_t\mathbf{x}_{t+1} = \mathbf{\Phi}\mathbf{x}_t$)

Observed data:
OLS regression of log real consumption on constant and time trend

residual $= z_{1t} = \widehat{C}_t$

Same for log of investment yields $\hat{I}_t$

Other data: GDP, real wages, GDP deflator, nominal interest rate

Treat $\hat{Q}_t, \hat{r}_{K,t}, \hat{a}_t^a$ as unobservable state equation:

$$\boldsymbol{\xi}_{t+1} = \mathbf{F}\boldsymbol{\xi}_t + \mathbf{v}_{t+1}$$

(came from solution to DSGE model)

observation equation:

$$\mathbf{z}_t = \mathbf{H}'\boldsymbol{\xi}_t$$

Let $\boldsymbol{\theta}$ = parameters of structural model

Use Kalman filter to evaluate likelihood function

$p(\mathbf{z}_1, \ldots, \mathbf{z}_T | \boldsymbol{\theta})$

prior $p(\boldsymbol{\theta})$

posterior $p(\boldsymbol{\theta}|\mathbf{Z}) \propto p(\boldsymbol{\theta})p(\mathbf{Z}|\boldsymbol{\theta})$

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we can sample from posterior using method such as Metropolis-Hastings

$\lambda(\boldsymbol{\theta}) = \log p(\boldsymbol{\theta}) + \log p(\mathbf{Z}|\boldsymbol{\theta})$

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(1) Find mode of posterior distribution using numerical optimization

$\boldsymbol{\theta}^* = \arg\max \lambda(\boldsymbol{\theta})$

(2) Find Hessian of posterior distribution

$\mathbf{H}(\boldsymbol{\theta}^*) = -\frac{\partial^2 \lambda(\boldsymbol{\theta})}{\partial\boldsymbol{\theta}\,\partial\boldsymbol{\theta}'}\Big|_{\boldsymbol{\theta}=\boldsymbol{\theta}^*}$

(3) Set $\boldsymbol{\theta}^{(j)}$ = arbitrary starting value for $j = 1$

(4) Generate $\tilde{\boldsymbol{\theta}}^{(j+1)} \sim N(\boldsymbol{\theta}^{(j)}, c\mathbf{H}(\boldsymbol{\theta}^*))$ for choice of c described below (could experiment to see if Student t works better)

(5) Generate $u^{(j+1)} \sim U[0,1]$

(6) Set $\boldsymbol{\theta}^{(j+1)} = \tilde{\boldsymbol{\theta}}^{(j+1)}$ if $u^{(j+1)} < \alpha^{(j+1)}$

$= \boldsymbol{\theta}^{(j)}$ if $u^{(j+1)} \geq \alpha^{(j+1)}$

$\alpha^{(j+1)} = \min\{1, \exp[\lambda(\boldsymbol{\theta}^{(j+1)}) - \lambda(\boldsymbol{\theta}^{(j)})]\}$

(7) Repeat steps (4)-(6) for
$j = 2, 3, \ldots, 10,000$

Values of $\boldsymbol{\theta}^{(j)}$ for $j \in \{5,001, \ldots, 10,000\}$ represent sample from $p(\boldsymbol{\theta}|\mathbf{Z})$

choose c so that have 20-30% acceptance at step (6)