

III. Linear state-space models

- A. State-space representation of a dynamic system
- B. Kalman filter
- C. Using the Kalman filter
- D. Bayesian analysis of linear state-space models
- E. Solutions to linear rational expectations models
 - 1. Problem statement

$$\mathbf{A}E_t\mathbf{y}_{t+1} = \mathbf{B}\mathbf{y}_t + \mathbf{C}\mathbf{x}_t$$

$$\mathbf{x}_{t+1} = \mathbf{\Phi}\mathbf{x}_t + \boldsymbol{\varepsilon}_{t+1}$$

\mathbf{x}_t exogenous

e.g., if $\mathbf{z}_t = (k \times 1)$

$$\begin{aligned}\mathbf{z}_t = & \phi_1 \mathbf{z}_{t-1} + \phi_2 \mathbf{z}_{t-2} + \dots \\ & + \phi_p \mathbf{z}_{t-p} + \mathbf{v}_t\end{aligned}$$

$$\mathbf{X}_t = \begin{bmatrix} \mathbf{z}_t \\ \mathbf{z}_{t-1} \\ \vdots \\ \mathbf{z}_{t+p+1} \end{bmatrix}$$

$(kp \times 1)$

$$\Phi = \begin{bmatrix} \phi_1 & \phi_2 & \cdots & \phi_{p-1} & \phi_p \\ \mathbf{I}_k & \mathbf{0} & \cdots & \mathbf{0} & \mathbf{0} \\ \vdots & \vdots & \cdots & \vdots & \vdots \\ \mathbf{0} & \mathbf{0} & \cdots & \mathbf{I}_k & \mathbf{0} \end{bmatrix}$$

$$\mathbf{y}_t = \begin{bmatrix} \mathbf{k}_t \\ (n_k \times 1) \\ \mathbf{d}_t \\ (n_d \times 1) \end{bmatrix}$$

$\mathbf{k}_t =$ predetermined

(chosen by agents at $t - 1$)

$$\mathbf{k}_t = h(\mathbf{x}_{t-1}, \mathbf{x}_{t-2}, \dots, \mathbf{x}_1, \mathbf{k}_0)$$

$$\mathbf{d}_t = m(\mathbf{x}_t, \mathbf{x}_{t-1}, \dots, \mathbf{x}_1, \mathbf{k}_0)$$

goal of solution method:

find $h(\cdot)$ and $m(\cdot)$

E. Solutions to linear rational expectations models

1. Problem statement

2. Blanchard-Kahn solution method

$$\mathbf{A}E_t\mathbf{y}_{t+1} = \mathbf{B}\mathbf{y}_t + \mathbf{C}\mathbf{x}_t$$

This method assumes \mathbf{A} is nonsingular.

Drawback: if original system involves

$E_t\mathbf{z}_{t+2}$ or \mathbf{z}_{t-2} , can be written in

canonical form using companion

form, but resulting \mathbf{A} may be singular.

$$\mathbf{A}E_t\mathbf{y}_{t+1} = \mathbf{B}\mathbf{y}_t + \mathbf{C}\mathbf{x}_t$$

Find Jordan form of $\mathbf{A}^{-1}\mathbf{B}$:

$$\mathbf{A}^{-1}\mathbf{B} = \mathbf{V}^{-1}\mathbf{J}\mathbf{V}$$

$$\mathbf{J} = \begin{bmatrix} \mathbf{J}_1 & \mathbf{0} & \cdots & \mathbf{0} \\ \mathbf{0} & \mathbf{J}_2 & \cdots & \mathbf{0} \\ \vdots & \vdots & \cdots & \vdots \\ \mathbf{0} & \mathbf{0} & \cdots & \mathbf{J}_s \end{bmatrix}$$

$$\mathbf{J}_i = \begin{bmatrix} \lambda_i & 1 & 0 & \cdots & 0 \\ 0 & \lambda_i & 1 & \cdots & 0 \\ 0 & 0 & \lambda_i & \cdots & 0 \\ \vdots & \vdots & \vdots & \cdots & \vdots \\ 0 & 0 & 0 & \cdots & \lambda_i \end{bmatrix}$$

Order eigenvalues λ_i such that first n_s are less than or equal to unity in modulus and next n_u are greater than unity in modulus. Assumption for unique stationary solution: $n_s = n_k$

$$\mathbf{J} = \begin{bmatrix} \mathbf{J}_s & \mathbf{0} \\ \mathbf{0} & \mathbf{J}_u \end{bmatrix}$$

diagonal elements of \mathbf{J}_s are

all ≤ 1 in modulus

diagonal elements of \mathbf{J}_u are

all > 1 in modulus

premultiply original system

$$\mathbf{A}E_t\mathbf{y}_{t+1} = \mathbf{B}\mathbf{y}_t + \mathbf{C}\mathbf{x}_t$$

by $\mathbf{V}\mathbf{A}^{-1}$

$$\mathbf{V}E_t\mathbf{y}_{t+1} = \mathbf{V}\mathbf{A}^{-1}\mathbf{B}\mathbf{y}_t + \mathbf{V}\mathbf{A}^{-1}\mathbf{C}\mathbf{x}_t$$

$$E_t\mathbf{V}\mathbf{y}_{t+1} = \mathbf{J}\mathbf{V}\mathbf{y}_t + \mathbf{C}^*\mathbf{x}_t$$

$$E_t \mathbf{V} \mathbf{y}_{t+1} = \mathbf{J} \mathbf{V} \mathbf{y}_t + \mathbf{C}^* \mathbf{x}_t$$

define

$$\begin{bmatrix} \mathbf{S}_t \\ (n_s \times 1) \\ \mathbf{u}_t \\ (n_d \times 1) \end{bmatrix} = \mathbf{V} \mathbf{y}_t$$

$$\begin{bmatrix} E_t \mathbf{S}_{t+1} \\ E_t \mathbf{u}_{t+1} \end{bmatrix} = \begin{bmatrix} \mathbf{J}_s \mathbf{S}_t \\ \mathbf{J}_u \mathbf{u}_t \end{bmatrix} + \begin{bmatrix} \mathbf{C}_{sx}^* \mathbf{x}_t \\ \mathbf{C}_{ux}^* \mathbf{x}_t \end{bmatrix}$$

$$E_t \mathbf{u}_{t+1} = \mathbf{J}_u \mathbf{u}_t + \mathbf{C}_{ux}^* \mathbf{x}_t$$

$$\mathbf{u}_t = \mathbf{J}_u^{-1} E_t \mathbf{u}_{t+1} - \mathbf{J}_u^{-1} \mathbf{C}_{ux}^* \mathbf{x}_t$$

$$\mathbf{u}_{t+1} = \mathbf{J}_u^{-1} E_{t+1} \mathbf{u}_{t+2} - \mathbf{J}_u^{-1} \mathbf{C}_{ux}^* \mathbf{x}_{t+1}$$

$$E_t \mathbf{u}_{t+1} = \mathbf{J}_u^{-1} E_t \mathbf{u}_{t+2} - \mathbf{J}_u^{-1} \mathbf{C}_{ux}^* E_t \mathbf{x}_{t+1}$$

$$\mathbf{u}_t = -\mathbf{J}_u^{-1} \mathbf{C}_{ux}^* \mathbf{x}_t - \mathbf{J}_u^{-2} \mathbf{C}_{ux}^* E_t \mathbf{x}_{t+1} + \mathbf{J}_u^{-2} E_t \mathbf{u}_{t+2}$$

$$\mathbf{u}_t = -\mathbf{J}_u^{-1} \sum_{h=0}^{\infty} \mathbf{J}_u^{-h} \mathbf{C}_{ux}^* E_t \mathbf{x}_{t+h}$$

$$\begin{bmatrix} E_t \mathbf{S}_{t+1} \\ E_t \mathbf{u}_{t+1} \end{bmatrix} = \begin{bmatrix} \mathbf{J}_s \mathbf{S}_t \\ \mathbf{J}_u \mathbf{u}_t \end{bmatrix} + \begin{bmatrix} \mathbf{C}_{sx}^* \mathbf{X}_t \\ \mathbf{C}_{ux}^* \mathbf{X}_t \end{bmatrix}$$

$$\begin{aligned} \mathbf{u}_t &= -\mathbf{J}_u^{-1} \sum_{h=0}^{\infty} \mathbf{J}_u^{-h} \mathbf{C}_{ux}^* E_t \mathbf{X}_{t+h} \\ &= -\mathbf{J}_u^{-1} \sum_{h=0}^{\infty} \mathbf{J}_u^{-h} \mathbf{C}_{ux}^* \Phi^h \mathbf{X}_t \\ &= \mathbf{H}_{ux} \mathbf{X}_t \end{aligned}$$

Note if

$$\mathbf{S} = \mathbf{J}_u^{-1} \sum_{h=0}^{\infty} \mathbf{J}_u^{-h} \mathbf{C}_{ux}^* \Phi^h$$

then

$$\mathbf{S} - \mathbf{J}_u^{-1} \mathbf{S} \Phi = \mathbf{J}_u^{-1} \mathbf{C}_{ux}^*$$

$$\text{vec}(\mathbf{S}) - (\Phi' \otimes \mathbf{J}_u^{-1}) \text{vec}(\mathbf{S}) = \text{vec}(\mathbf{J}_u^{-1} \mathbf{C}_{ux}^*)$$

$$\text{vec}(\mathbf{S}) = [\mathbf{I}_{n_u^2} - \Phi' \otimes \mathbf{J}_u^{-1}]^{-1} \text{vec}(\mathbf{J}_u^{-1} \mathbf{C}_{ux}^*)$$

$$\mathbf{u}_t = -\mathbf{J}_u^{-1} \sum_{h=0}^{\infty} \mathbf{J}_u^{-h} \mathbf{C}_{ux}^* \Phi^h \mathbf{x}_t = \mathbf{H}_{ux} \mathbf{x}_t$$

$$\mathbf{H}_{ux} = -[\mathbf{I}_{n_u^2} - \Phi' \otimes \mathbf{J}_u^{-1}]^{-1} \text{vec}(\mathbf{J}_u^{-1} \mathbf{C}_{ux}^*)$$

$$\begin{bmatrix} \mathbf{s}_t \\ \mathbf{u}_t \end{bmatrix} = \mathbf{V} \mathbf{y}_t = \mathbf{V} \begin{bmatrix} \mathbf{k}_t \\ \mathbf{d}_t \end{bmatrix}$$

$$\mathbf{s}_t = \mathbf{V}_{sk} \mathbf{k}_t + \mathbf{V}_{sd} \mathbf{d}_t$$

$$\mathbf{u}_t = \mathbf{V}_{uk} \mathbf{k}_t + \mathbf{V}_{ud} \mathbf{d}_t$$

$$\mathbf{u}_t = \mathbf{V}_{uk} \mathbf{k}_t + \mathbf{V}_{ud} \mathbf{d}_t$$

suppose that \mathbf{V}_{ud}^{-1} exists

(required to be able to choose \mathbf{d}_t so as to eliminate unstable eigenvalues)

$$\mathbf{d}_t = \mathbf{V}_{ud}^{-1} \mathbf{u}_t - \mathbf{V}_{ud}^{-1} \mathbf{V}_{uk} \mathbf{k}_t$$

$$\mathbf{d}_t = \mathbf{V}_{ud}^{-1} \mathbf{u}_t - \mathbf{V}_{ud}^{-1} \mathbf{V}_{uk} \mathbf{k}_t$$

$$\mathbf{s}_t = \mathbf{V}_{sk} \mathbf{k}_t + \mathbf{V}_{sd} \mathbf{d}_t$$

$$\mathbf{s}_t = \mathbf{V}_{sk} \mathbf{k}_t + \mathbf{V}_{sd} [\mathbf{V}_{ud}^{-1} \mathbf{u}_t - \mathbf{V}_{ud}^{-1} \mathbf{V}_{uk} \mathbf{k}_t]$$

Also define $\mathbf{R} = \mathbf{V}^{-1}$

$$\begin{bmatrix} \mathbf{s}_t \\ \mathbf{u}_t \end{bmatrix} = \mathbf{V} \begin{bmatrix} \mathbf{k}_t \\ \mathbf{d}_t \end{bmatrix}$$

$$\begin{bmatrix} \mathbf{k}_t \\ \mathbf{d}_t \end{bmatrix} = \mathbf{R} \begin{bmatrix} \mathbf{s}_t \\ \mathbf{u}_t \end{bmatrix}$$

$$\mathbf{k}_t = \mathbf{R}_{ks} \mathbf{s}_t + \mathbf{R}_{ku} \mathbf{u}_t$$

$$\mathbf{k}_t = \mathbf{R}_{ks}\mathbf{s}_t + \mathbf{R}_{ku}\mathbf{u}_t$$

$$\mathbf{k}_{t+1} = \mathbf{R}_{ks}\mathbf{s}_{t+1} + \mathbf{R}_{ku}\mathbf{u}_{t+1}$$

$$\begin{aligned} E_t\mathbf{k}_{t+1} &= \mathbf{R}_{ks}E_t\mathbf{s}_{t+1} + \mathbf{R}_{ku}E_t\mathbf{u}_{t+1} \\ &= \mathbf{k}_{t+1} \end{aligned}$$

(b/c \mathbf{k}_{t+1} is determined at t)

Recall

$$\begin{bmatrix} E_t \mathbf{s}_{t+1} \\ E_t \mathbf{u}_{t+1} \end{bmatrix} = \begin{bmatrix} \mathbf{J}_s \mathbf{s}_t \\ \mathbf{J}_u \mathbf{u}_t \end{bmatrix} + \begin{bmatrix} \mathbf{C}_{sx}^* \mathbf{x}_t \\ \mathbf{C}_{ux}^* \mathbf{x}_t \end{bmatrix}$$

$$\mathbf{k}_{t+1} = \mathbf{R}_{ks} E_t \mathbf{s}_{t+1} + \mathbf{R}_{ku} E_t \mathbf{u}_{t+1}$$

$$\begin{aligned} \mathbf{k}_{t+1} &= \mathbf{R}_{ks} (\mathbf{J}_s \mathbf{s}_t + \mathbf{C}_{sx}^* \mathbf{x}_t) \\ &\quad + \mathbf{R}_{ku} (\mathbf{J}_u \mathbf{u}_t + \mathbf{C}_{ux}^* \mathbf{x}_t) \end{aligned}$$

$$\mathbf{k}_{t+1} = \mathbf{R}_{ks}(\mathbf{J}_s \mathbf{s}_t + \mathbf{C}_{sx}^* \mathbf{x}_t) \\ + \mathbf{R}_{ku}(\mathbf{J}_u \mathbf{u}_t + \mathbf{C}_{ux}^* \mathbf{x}_t)$$

Recall also

$$\mathbf{s}_t = \mathbf{V}_{sk} \mathbf{k}_t + \mathbf{V}_{sd} [\mathbf{V}_{ud}^{-1} \mathbf{u}_t - \mathbf{V}_{ud}^{-1} \mathbf{V}_{uk} \mathbf{k}_t]$$

so

$$\mathbf{k}_{t+1} = \mathbf{R}_{ks} \mathbf{J}_s \{ \mathbf{V}_{sk} \mathbf{k}_t + \mathbf{V}_{sd} [\mathbf{V}_{ud}^{-1} \mathbf{u}_t - \mathbf{V}_{ud}^{-1} \mathbf{V}_{uk} \mathbf{k}_t] \} \\ + \mathbf{R}_{ks} \mathbf{C}_{sx}^* \mathbf{x}_t + \mathbf{R}_{ku} (\mathbf{J}_u \mathbf{u}_t + \mathbf{C}_{ux}^* \mathbf{x}_t)$$

Finally, since

$$\mathbf{u}_t = \mathbf{H}_{ux} \mathbf{x}_t$$

$$\mathbf{H}_{ux} = -[\mathbf{I}_{n_u^2} - \Phi' \otimes \mathbf{J}_u^{-1}]^{-1} \text{vec}(\mathbf{J}_u^{-1} \mathbf{C}_{ux}^*)$$

$$\mathbf{k}_{t+1} = \mathbf{H}_{kk} \mathbf{k}_t + \mathbf{H}_{kx} \mathbf{x}_t$$

$$\mathbf{H}_{kk} = \mathbf{R}_{ks} \mathbf{J}_s \mathbf{V}_{sk} - \mathbf{R}_{ks} \mathbf{J}_s \mathbf{V}_{sd} \mathbf{V}_{ud}^{-1} \mathbf{V}_{uk}$$

$$\mathbf{H}_{kx} = \mathbf{R}_{ks} \mathbf{J}_s \mathbf{V}_{sd} \mathbf{V}_{ud}^{-1} \mathbf{H}_{ux} + \mathbf{R}_{ks} \mathbf{C}_{sx}^*$$

$$+ \mathbf{R}_{ku} \mathbf{J}_u \mathbf{H}_{ux} + \mathbf{R}_{ku} \mathbf{C}_{ux}^*$$

From our earlier expression

$$\mathbf{d}_t = \mathbf{V}_{ud}^{-1} \mathbf{u}_t - \mathbf{V}_{ud}^{-1} \mathbf{V}_{uk} \mathbf{k}_t$$

we likewise have

$$\mathbf{d}_t = \mathbf{V}_{ud}^{-1} \mathbf{H}_{ux} \mathbf{x}_t - \mathbf{V}_{ud}^{-1} \mathbf{V}_{uk} \mathbf{k}_t$$

Conclusion: the system

$$\mathbf{A}E_t\mathbf{y}_{t+1} = \mathbf{B}\mathbf{y}_t + \mathbf{C}\mathbf{x}_t$$

$$\mathbf{x}_{t+1} = \mathbf{\Phi}\mathbf{x}_t + \boldsymbol{\varepsilon}_{t+1}$$

$$\mathbf{y}_t = \begin{bmatrix} \mathbf{k}_t \\ \mathbf{d}_t \end{bmatrix}$$

has the solution

$$\mathbf{k}_{t+1} = \mathbf{H}_{kk}\mathbf{k}_t + \mathbf{H}_{kx}\mathbf{x}_t$$

$$\mathbf{d}_t = \mathbf{V}_{ud}^{-1}\mathbf{H}_{ux}\mathbf{x}_t - \mathbf{V}_{ud}^{-1}\mathbf{V}_{uk}\mathbf{k}_t$$

$$\mathbf{x}_{t+1} = \mathbf{\Phi}\mathbf{x}_t + \mathbf{\varepsilon}_{t+1}$$

which is a state-space system

$$\text{with } \xi_{t+1} = (\mathbf{x}'_{t+1}, \mathbf{k}'_{t+1})'$$

E. Solutions to linear rational expectations models

1. Problem statement

2. Blanchard-Kahn solution method

3. Klein solution method

$$\mathbf{A}E_t\mathbf{y}_{t+1} = \mathbf{B}\mathbf{y}_t + \mathbf{C}\mathbf{x}_t$$

$$\mathbf{x}_{t+1} = \mathbf{\Phi}\mathbf{x}_t + \boldsymbol{\varepsilon}_{t+1}$$

More general case:

\mathbf{A} is singular

As long as there exists a
(possibly complex) scalar z
such that $|\mathbf{A}z - \mathbf{B}| \neq 0$
then there exist (possibly complex)
matrices \mathbf{Q} and \mathbf{Z} such that

$$(i) \mathbf{Q}^H \mathbf{Q} = \mathbf{I}_n$$

where \mathbf{Q}^H means transpose
and take complex conjugates

(ii) $\mathbf{Z}^H \mathbf{Z} = \mathbf{I}_n$

(iii) $\mathbf{QAZ} = \mathbf{S}$ is upper triangular

(iv) $\mathbf{QBZ} = \mathbf{T}$ is upper triangular

(v) variables can be ordered

so that t_{ii}/s_{ii} are increasing

in modulus (with any zero s_{ii}

appearing last)

Called complex generalized
Schur form

Assume the first n_s values of
 t_{ii}/s_{ii} are < 1 and last n_u are > 1

premultiply original system

$$\mathbf{A}E_t\mathbf{y}_{t+1} = \mathbf{B}\mathbf{y}_t + \mathbf{C}\mathbf{x}_t$$

by \mathbf{Q}

$$E_t\mathbf{QAZZ}^H\mathbf{y}_{t+1} = \mathbf{QBZZ}^H\mathbf{y}_t + \mathbf{QC}\mathbf{x}_t$$

$$E_t \mathbf{QAZZ}^H \mathbf{y}_{t+1} = \mathbf{QBZZ}^H \mathbf{y}_t + \mathbf{QCx}_t$$

$$\mathbf{QAZ} = \mathbf{S} \quad \mathbf{QBZ} = \mathbf{T}$$

$$E_t \mathbf{SZ}^H \mathbf{y}_{t+1} = \mathbf{TZ}^H \mathbf{y}_t + \mathbf{QCx}_t$$

$$E_t \mathbf{S} \mathbf{Z}^H \mathbf{y}_{t+1} = \mathbf{T} \mathbf{Z}^H \mathbf{y}_t + \mathbf{Q} \mathbf{C} \mathbf{x}_t$$

Define

$$\begin{bmatrix} \mathbf{s}_t \\ \mathbf{u}_t \end{bmatrix} = \mathbf{Z}^H \mathbf{y}_t$$

$$\mathbf{S} \begin{bmatrix} E_t \mathbf{s}_{t+1} \\ E_t \mathbf{u}_{t+1} \end{bmatrix} = \mathbf{T} \begin{bmatrix} \mathbf{s}_t \\ \mathbf{u}_t \end{bmatrix} + \mathbf{C}^* \mathbf{x}_t$$

$$\begin{aligned}
 & \begin{bmatrix} \mathbf{S}_{11} & \mathbf{S}_{12} \\ \mathbf{0} & \mathbf{S}_{22} \end{bmatrix} \begin{bmatrix} E_t \mathbf{s}_{t+1} \\ E_t \mathbf{u}_{t+1} \end{bmatrix} \\
 &= \begin{bmatrix} \mathbf{T}_{11} & \mathbf{T}_{12} \\ \mathbf{0} & \mathbf{T}_{22} \end{bmatrix} \begin{bmatrix} \mathbf{s}_t \\ \mathbf{u}_t \end{bmatrix} \\
 &+ \begin{bmatrix} \mathbf{C}_{sx}^* \mathbf{X}_t \\ \mathbf{C}_{ux}^* \mathbf{X}_t \end{bmatrix}
 \end{aligned}$$

where \mathbf{T}_{22} is invertible by construction

$$\mathbf{S}_{22} \mathbf{E}_t \mathbf{u}_{t+1} = \mathbf{T}_{22} \mathbf{u}_t + \mathbf{C}_{ux}^* \mathbf{x}_t$$

$$\begin{aligned} \mathbf{u}_t &= -\mathbf{T}_{22}^{-1} \sum_{h=0}^{\infty} [\mathbf{T}_{22}^{-1} \mathbf{S}_{22}]^h \mathbf{C}_{ux}^* \mathbf{E}_t \mathbf{x}_{t+h} \\ &= -\mathbf{T}_{22}^{-1} \sum_{h=0}^{\infty} [\mathbf{T}_{22}^{-1} \mathbf{S}_{22}]^h \mathbf{C}_{ux}^* \mathbf{\Phi}^h \mathbf{x}_t \\ &= \mathbf{H}_{ux} \mathbf{x}_t \end{aligned}$$

$$\mathbf{H}_{ux} = [(\mathbf{\Phi}' \otimes \mathbf{S}_{22}) - (\mathbf{I}_{n_x} \otimes \mathbf{T}_{22})]^{-1} \text{vec}(\mathbf{C}_{ux}^*)$$

Can then use parallel calculations
to those for Blanchard-Kahn to
arrive at

$$\mathbf{k}_{t+1} = \mathbf{H}_{kk}\mathbf{k}_t + \mathbf{H}_{kx}\mathbf{X}_t$$

$$\mathbf{d}_t = \mathbf{V}_{ud}^{-1}\mathbf{H}_{ux}\mathbf{X}_t - \mathbf{V}_{ud}^{-1}\mathbf{V}_{uk}\mathbf{k}_t$$

$$\mathbf{X}_{t+1} = \mathbf{\Phi}\mathbf{X}_t + \boldsymbol{\varepsilon}_{t+1}$$