- Practice exams with answers available on course web page at http://econweb.ucsd.edu/~jhamilto/Econ226.html

II. Vector autoregressions

A. Introduction
B. Normal-Wishart priors for VAR’s
C. Bayesian analysis of structural VAR’s
D. Identification using inequality constraints
E. Integrating VARs with dynamic general equilibrium models

Del Negro and Schorfheide:

$X_t(j)$ = output of firm j

X_t = output of representative firm in symmetric equilibrium

$h_t(j)$ = labor hired by firm j

A_t = productivity

$$X_t(j) = A_t h_t(j)$$

$C_t =$ consumption of representative household

$G_t =$ government spending

$$X_t = C_t + G_t$$

$$G_t = \varsigma_t X_t$$

Sources of randomness:

(1) technology shock

$$\log A_t = \log \gamma + \log A_{t-1} + \tilde{z}_t$$

$$\tilde{z}_t = \rho_z \tilde{z}_{t-1} + u_{zt}$$

(2) government spending shock

$$\tilde{g}_t = g^* - \log(1 - \varsigma_t)$$

$$\tilde{g}_t = \rho_g \tilde{g}_{t-1} + u_{gt}$$

(3) Fed policy shock (Taylor rule)

$$\tilde{r}_t = \rho_R \tilde{r}_{t-1} + (1-\rho_R)(\psi_1 \tilde{\pi}_t + \psi_2 \tilde{x}_t)$$
$$+ u_{ft}$$

Will use these three shocks to explain output growth, fed funds rate, and inflation rate

$$U(C_t, M_t/P_t, h_t) = \frac{(C_t/A_t)^{1-\tau}}{1-\tau} + \chi \log(M_t/P_t) - h_t$$

Euler equation:

$$U_C(C_t, M_t/P_t, h_t) = \beta E_t\{U_C(C_{t+1}, M_{t+1}/P_{t+1}, h_{t+1}) \times (1+i_t)/(1+\pi_{t+1})\}$$

Substitute in equilibrium condition

$$C_t = (1 - \varsigma_t) X_t$$

and log linearize:

$$\tilde{x}_t = E_t \tilde{x}_{t+1} - \tau^{-1} [\tilde{r}_t - E_t \tilde{\pi}_{t+1}]$$

$$+ (1 - \rho_G) \tilde{g}_t + (\rho_z / \tau) \tilde{z}_t$$

Euler equation:

$$\tilde{x}_t = E_t\tilde{x}_{t+1} - \tau^{-1}[\tilde{r}_t - E_t\tilde{\pi}_{t+1}] + (1-\rho_G)\tilde{g}_t + (\rho_z/\tau)\tilde{z}_t$$

Firms face quadratic costs in adjusting prices:

$$\tilde{\pi}_t = \beta E_t\tilde{\pi}_{t+1} + \kappa(\tilde{x}_t - \tilde{g}_t)$$

θ vector of structural parameters

γ average productivity growth

β personal discount rate

κ slope of Phillips curve (function of adjustment costs and demand elasticity)

τ coefficient of relative risk aversion

ψ_1, ψ_2 coefficients on inflation and output in Taylor rule

ρ_g, ρ_z serial correlation of govt. spending and productivity shocks

$\sigma_f, \sigma_g, \sigma_z$ standard deviations of structural shocks

Observable data:

y_{1t} quarterly real GDP growth

y_{2t} quarterly CPI inflation

y_{3t} average fed funds rate over quarter

$$
\begin{aligned}
y_{1t} &= \log(\gamma) + \Delta\tilde{x}_t + \tilde{z}_t \\
y_{2t} &= \log(\pi^*) + \tilde{\pi}_t \\
y_{3t} &= \log(\beta/\gamma) + \log(\pi^*) + \tilde{r}_t
\end{aligned}
$$

Given $\boldsymbol{\theta}$, can use numerical methods to calculate model's predictions for

$$E(\mathbf{y}_{\tilde{t}}\mathbf{y}_{\tilde{t}}') = \mathbf{C}(\boldsymbol{\theta})$$

$$E(\mathbf{x}_{\tilde{t}}\mathbf{x}_{\tilde{t}}') = \mathbf{A}(\boldsymbol{\theta})$$

$$\mathbf{x}_{\tilde{t}} = (1, \mathbf{y}_{\tilde{t}-1}', \mathbf{y}_{\tilde{t}-2}', \ldots, \mathbf{y}_{\tilde{t}-p}')'$$

$$E(\mathbf{x}_{\tilde{t}}\mathbf{y}_{\tilde{t}}') = \mathbf{B}(\boldsymbol{\theta})$$

prior for $\mathbf{\Omega}$:

$$\mathbf{\Omega}^{-1}|\theta \sim W(\tilde{T}, \mathbf{\Lambda}(\theta))$$

$\tilde{T}$ number of observations we want to treat theoretical model as equivalent to

$$\mathbf{\Lambda}(\theta) = \tilde{T}\left[\mathbf{C}(\theta) - \mathbf{B}(\theta)'\mathbf{A}(\theta)^{-1}\mathbf{B}(\theta)\right]$$

prior for $\boldsymbol{\gamma}$:

$$\boldsymbol{\gamma}|\boldsymbol{\Omega},\boldsymbol{\theta} \sim N(\mathbf{m}(\boldsymbol{\theta}),\boldsymbol{\Omega}\otimes\mathbf{M}(\boldsymbol{\theta}))$$

$$\mathbf{m}(\boldsymbol{\theta}) = \mathrm{vec}\left[\mathbf{A}(\boldsymbol{\theta})^{-1}\mathbf{B}(\boldsymbol{\theta})\right]$$

$$\mathbf{M}(\boldsymbol{\theta}) = \left[\tilde{T}\mathbf{A}(\boldsymbol{\theta})\right]^{-1}$$

prior for θ:

τ Gamma with mean 2

standard deviation 0.5

ψ_1 Gamma with mean 1.5

standard deviation 0.25

prior for all parameters:

$$f(\boldsymbol{\theta})f(\boldsymbol{\Omega}^{-1}|\boldsymbol{\theta})f(\boldsymbol{\gamma}|\boldsymbol{\Omega},\boldsymbol{\theta})$$

Have analytical expressions for

$f(\mathbf{Y}|\boldsymbol{\gamma}, \boldsymbol{\Omega})$ from likelihood

(same as $f(\mathbf{Y}|\boldsymbol{\gamma}, \boldsymbol{\Omega}, \boldsymbol{\theta})$)

$f(\boldsymbol{\gamma}, \boldsymbol{\Omega}|\boldsymbol{\theta})$ from prior

$f(\boldsymbol{\gamma}, \boldsymbol{\Omega}|\boldsymbol{\theta}, \mathbf{Y})$ from posterior

From these we could calculate the value of

$$f(\mathbf{Y}|\boldsymbol{\theta}) = \frac{f(\mathbf{Y}|\boldsymbol{\gamma}, \boldsymbol{\Omega}, \boldsymbol{\theta})f(\boldsymbol{\gamma}, \boldsymbol{\Omega}|\boldsymbol{\theta})}{f(\boldsymbol{\gamma}, \boldsymbol{\Omega}|\mathbf{Y}, \boldsymbol{\theta})}$$

Using this and prior $f(\boldsymbol{\theta})$, we can use numerical methods (Metropolis-Hastings or importance sampling) to sample from

$$f(\boldsymbol{\theta}|\mathbf{Y}) \propto f(\mathbf{Y}|\boldsymbol{\theta})f(\boldsymbol{\theta})$$

to get inference about true structural parameters

90% posterior confidence

$$\tau \in (1.3, 2.8)$$

$$\psi_1 \in (1.0, 1.6)$$

By sampling $\boldsymbol{\theta}^{(m)}$ from $f(\boldsymbol{\theta}|\mathbf{Y})$ and then $\boldsymbol{\Omega}^{(m)}$ from $f(\boldsymbol{\Omega}|\boldsymbol{\theta}^{(m)},\mathbf{Y})$ and $\boldsymbol{\gamma}^{(m)}$ from $f(\boldsymbol{\gamma}|\boldsymbol{\Omega}^{(m)},\boldsymbol{\theta}^{(m)},\mathbf{Y})$, we can generate posterior distribution of VAR parameters

Identifying impulse-response function using structural model

$$\begin{bmatrix} u_{gt} \\ u_{zt} \\ u_{ft} \end{bmatrix} = \begin{bmatrix} \sigma_g v_{gt} \\ \sigma_z v_{zt} \\ \sigma_f v_{ft} \end{bmatrix}$$

$$E(\mathbf{v}_t \mathbf{v}_t') = \mathbf{I}_3$$

$$\mathbf{A}_0 \mathbf{y}_t = \mathbf{a} + \mathbf{A}_1 \mathbf{y}_{t-1} + \mathbf{A}_2 \mathbf{y}_{t-2} + \cdots + \mathbf{A}_p \mathbf{y}_{t-p} + \mathbf{v}_t$$

Can calculate from structural model

$$\mathbf{A}_0 (\boldsymbol{\theta})^{-1} = \frac{\partial \mathbf{y}_t}{\partial \mathbf{v}_t'}$$

One option:

$$\frac{\partial \mathbf{y}_{t+s}}{\partial \mathbf{v}_t^{'}} = \frac{\partial \mathbf{y}_{t+s}}{\partial \boldsymbol{\varepsilon}_t^{'}} \frac{\partial \boldsymbol{\varepsilon}_t}{\partial \mathbf{v}_t^{'}}$$

$$= \boldsymbol{\Psi}_s^{(m)} \mathbf{A}_0 (\boldsymbol{\theta}^{(m)})^{-1}$$

Problem:

$$\boldsymbol{\Omega}^{(m)} \neq \mathbf{A}_0(\boldsymbol{\theta}^{(m)})^{-1}[\mathbf{A}_0(\boldsymbol{\theta}^{(m)})^{-1}]'$$

$\boldsymbol{\Psi}^{(m)}\mathbf{A}_0(\boldsymbol{\theta}^{(m)})^{-1}$ OK for impulse-response, but variance shares won’t add up

Another option: QR factorization

Proposition: Any $(n \times k)$ matrix $\mathbf{C}$ can be written as

$$\underset{(n\times k)}{\mathbf{C}} = \underset{(n\times n)}{\mathbf{Q}} \underset{(n\times k)}{\mathbf{R}}$$

where $\mathbf{Q}'\mathbf{Q} = \mathbf{I}_n$

and $\mathbf{R}$ is upper triangular

How to find: qr command in Gauss or Matlab

(1) Find QR factorization of

$$[\mathbf{A}_0(\boldsymbol{\theta}^{(m)})^{-1}]' = \mathbf{Q}(\boldsymbol{\theta}^{(m)})\mathbf{R}(\boldsymbol{\theta}^{(m)})$$

$$\mathbf{A}_0(\boldsymbol{\theta}^{(m)})^{-1} = [\mathbf{R}(\boldsymbol{\theta}^{(m)})]'[\mathbf{Q}(\boldsymbol{\theta}^{(m)})]'$$

$[\mathbf{R}(\boldsymbol{\theta}^{(m)})]'$ is thus lower triangular

(2) Find Cholesky factorization of

$$\boldsymbol{\Omega}^{(m)} = \mathbf{P}^{(m)}\left[\mathbf{P}^{(m)}\right]'$$

so $\mathbf{P}^{(m)}$ is lower triangular

Assumption: $\mathbf{P}^{(m)} \simeq [\mathbf{R}(\boldsymbol{\theta}^{(m)})]'$

(3) Use

$$\boldsymbol{\varepsilon}_t = \mathbf{P}^{(m)}[\mathbf{Q}(\boldsymbol{\theta}^{(m)})]'\mathbf{v}_t$$

$$E(\boldsymbol{\varepsilon}_t\boldsymbol{\varepsilon}_t')$$

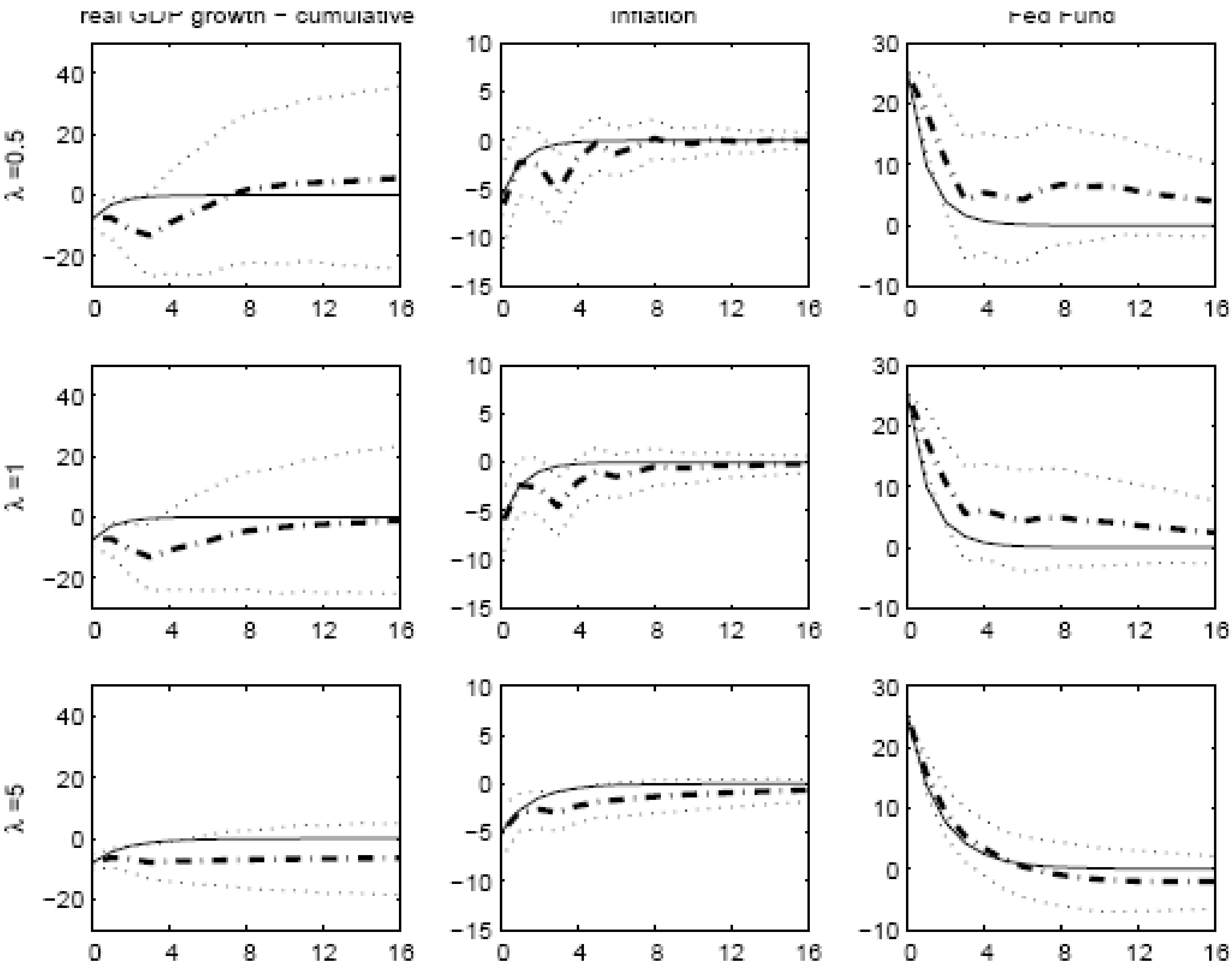
$$= \mathbf{P}^{(m)}[\mathbf{Q}(\boldsymbol{\theta}^{(m)})]'\mathbf{Q}(\boldsymbol{\theta}^{(m)})\mathbf{P}^{(m)}$$

$$= \mathbf{P}^{(m)}\mathbf{I}_n\left[\mathbf{P}^{(m)}\right]'$$

$$= \boldsymbol{\Omega}^{(m)}$$

In other words,

$$\frac{\partial \mathbf{y}_{t+s}}{\partial \mathbf{v}_t'} = \boldsymbol{\Psi}_s^{(m)} \mathbf{P}^{(m)} [\mathbf{Q}(\boldsymbol{\theta}^{(m)})]'$$



real GDP growth − cumulative
Inflation
Fed Fund
λ =0.5
λ =1
λ =5

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F. Selecting priors for DSGEs

Traditional approach:

- set priors for each parameter independently
- just augment with another independent prior when a new parameter is introduced

Model 1:

$$y_t = c + \varepsilon_t$$

$$\varepsilon_t \sim N(0, \sigma_\varepsilon^2)$$

Prior 1:

$$c \sim N(m, \lambda^2)$$

implies:

$$y_t \sim N(m, \sigma_\varepsilon^2 + \lambda^2)$$

Model 2:

$$y_t = c + by_{t-1} + \varepsilon_t$$

Suppose we keep specification of ε_t and prior for c the same, and just add independent prior for b:

$$b \sim U[0, \xi] \quad 0 < \xi < 1$$

Implies:

$$y_t|b \sim N\left(\frac{m}{1-b}, \frac{\sigma_\varepsilon^2+\lambda^2}{(1-b^2)}\right)$$

$$E(y_t) = E_b[E(y_t|b)] \;\; > m$$

Var$(y_t) > \sigma_\varepsilon^2 + \lambda^2$

Better approach: for Model 2

use the prior

$$c|b \sim N(m(1-b), \lambda^2(1-b^2))$$

Separate DSGE parameters

into 3 groups:

(1) steady-state parameters

α = production function parameter

$$Y_t(i) = A_t K_t(i)^{\alpha} L_t(i)^{1-\alpha}$$

λ_f = mark-up parameter

$$Y_t(i) = \left(\frac{P_t(i)}{P_t}\right)^{-(1+\lambda_f)/\lambda_f} Y_t$$

labor share = $\frac{1-\alpha}{1+\lambda_f}$

Del Negro and Schorfheide:

Represent prior as fictitious measurement:

$$\frac{1-\alpha}{1+\lambda_f} = 0.57 + v$$

$$v \sim N(0, 0.02)$$

Can also supplement with much less informative, separate priors:

$$\alpha \sim U[0,1]$$

$$\lambda_f \sim \Gamma(0.15, 0.10)$$

Other steady-state parameters include:

- β (discount rate)
- δ (depreciation rate)
- γ (technological growth rate)
- π^* (Fed's inflation target)
- g^* (government share)

Fictitious observations:

labor share

government share

investment-capital ratio

capital-output ratio

$\mathbf{s} =$ fictitious observations

$\mathbf{s}$ = fictitious observations

$\mathcal{L}(\mathbf{s}|\theta_{ss})$ = implied likelihood

$p(\boldsymbol{\theta}_{ss})$ = uninformative prior

$\hat{p}(\boldsymbol{\theta}_{ss}) \propto \mathcal{L}(\mathbf{s}|\theta_{ss})p(\boldsymbol{\theta}_{ss})$

is proposed prior on $\boldsymbol{\theta}_{ss}$

Separate DSGE parameters
into 3 groups:
(1) steady-state parameters
(2) endogenous parameters

$$a_t = \log A_t$$

$$a_t = (1 - \rho_a)\gamma + \rho_a a_{t-1} + \sigma_a \varepsilon_{at}$$

Usual approach:

pick ρ_a and σ_a to match autocorrelation and variance of output

Let $\mathbf{y}_t$ = vector of covariance-stationary variables for which DSGE has implications

$\Gamma^*_{\mathbf{yy}}$ = prior belief for $E(\mathbf{y}_t \mathbf{y}_t')$

$\boldsymbol{\Gamma}_{\mathbf{yy}}(\boldsymbol{\theta})$ = model implications

$$\mathbf{x}_t = (1, \mathbf{y}'_{t-1}, \mathbf{y}'_{t-2}, \ldots, \mathbf{y}'_{t-p})'$$

$\Gamma^*_{\mathbf{xy}}$ = prior belief for $E(\mathbf{x}_t \mathbf{y}'_t)$

$\Phi(\theta)$ = model implications for VAR coefficients

$$\Phi(\theta) = \Gamma_{\mathbf{xx}}(\theta)^{-1}\Gamma_{\mathbf{xy}}(\theta)$$

$\Omega(\theta)$ = model implications for VAR residual matrix

$$\Omega(\theta) = \Gamma_{\mathbf{yy}}(\theta) - \Gamma_{\mathbf{yx}}(\theta)\Gamma_{\mathbf{xx}}^{-1}(\theta)\Gamma_{\mathbf{xy}}(\theta)$$

T^* = weight on prior (number of equivalent observations)

$$p(\boldsymbol{\theta}_{\text{endo}}) \propto |\boldsymbol{\Omega}(\boldsymbol{\theta})|^{-(T^*+n+1)/2} \times$$
$$\exp\{-\frac{T^*}{2}\text{tr}[\boldsymbol{\Omega}(\boldsymbol{\theta})^{-1} \times$$
$$(\boldsymbol{\Gamma}^*_{\mathbf{yy}} - 2\boldsymbol{\Phi}(\boldsymbol{\theta})'\boldsymbol{\Gamma}^*_{\mathbf{xy}} + \boldsymbol{\Phi}(\boldsymbol{\theta})'\boldsymbol{\Gamma}^*_{\mathbf{xx}}\boldsymbol{\Phi}(\boldsymbol{\theta})]\}$$

Separate DSGE parameters
into 3 groups:
(1) steady-state parameters
(2) endogenous parameters
(3) exogenous parameters

ζ_p = fraction of firms who are not able toreoptimize their price

ℓ_p = fraction of firms who mechanicaly adjust price

Taylor Rule parameters (ρ_R, ψ_1, ψ_2) followed by Fed to set interest rate R_t:

$$R_t = \rho_R R_{t-1} + (1 - \rho_R)(\psi_1 \pi_t + \psi_2 y_t) + \sigma_R \varepsilon_{Rt}$$

Del Negro and Schorfheide use standard priors for this third set.

Examine implications of how you form the prior for posterior inference.

Posterior distribution for wage inflexibility fraction ζ_w:

usual priors:

mean: 0.52

90% confidence: [0.16, 0.81]

DS priors:

mean: 0.19

90% confidence: [0.10, 0.29]