Econ 220B Final Exam
Winter 2017

DIRECTIONS: No books or notes of any kind are allowed. Answer all questions on separate paper. 250 points are possible on this exam.

1.) (10 points each, 50 points total) The least-squares regression model $\mathbf{y} = \mathbf{X}\boldsymbol{\beta} + \boldsymbol{\varepsilon}$ makes a lot of assumptions, including:
(i) $\mathbf{X}'\mathbf{X}$ is of full rank k
(ii) $E(\varepsilon_t^2) = \sigma^2$
(iii) $\text{plim}(\mathbf{X}'\mathbf{X}/T) = \mathbf{Q}$ of full rank k
(iv) $T^{-1/2}\sum_{t=1}^{T}\mathbf{x}_t\varepsilon_t \xrightarrow{L} N(\mathbf{0},\sigma^2\mathbf{Q})$.

In addition, we often make one or more of the following assumptions:
(v) $E(\boldsymbol{\varepsilon}\,|\mathbf{X}) = \mathbf{0}$;
(vi) $E(\boldsymbol{\varepsilon}\boldsymbol{\varepsilon}'\,|\mathbf{X}) = \sigma^2\mathbf{I}_T$;
(vii) $\boldsymbol{\varepsilon}\,|\mathbf{X} \sim N(\mathbf{0},\sigma^2\mathbf{I}_T)$.
(viii) $\mathbf{x}_t$ is strictly deterministic

Questions (a)-(e) below involve certain statements about properties of the least-squares regression coefficient $\mathbf{b} = (\mathbf{X}'\mathbf{X})^{-1}\mathbf{X}'\mathbf{y}$ and variance estimate $s^2 = (T-k)^{-1}\mathbf{e}'\mathbf{e}$ for $\mathbf{e} = \mathbf{y} - \mathbf{X}\mathbf{b}$. In each of the questions, you can assume that (i)-(iv) are satisfied. The question is, which further assumptions besides (i)-(iv) would you need in order to prove the stated result? For example, if you think assumptions (i), (ii), and (v) alone are sufficient to prove that $E(\mathbf{b}) = \boldsymbol{\beta}$, then your answer to part (a) should be "v". If you think that (i), (iii), (v) and (vi) are needed to prove that $E(\mathbf{b}) = \boldsymbol{\beta}$, then your answer to part (a) should be "v and vi". Note that you should not give any proofs or arguments– your answer in each case should be some combination of the numerals "v", "vi", "vii", or "viii".
a.) $E(\mathbf{b}) = \boldsymbol{\beta}$
b.) $\mathbf{b}$ has minimum variance among the class of unbiased estimators of $\boldsymbol{\beta}$
c.) $E[(\mathbf{b}-\boldsymbol{\beta})(\mathbf{b}-\boldsymbol{\beta})'\,|\mathbf{X}] = \sigma^2(\mathbf{X}'\mathbf{X})^{-1}$
d.) $h(\mathbf{b}) \xrightarrow{p} h(\boldsymbol{\beta})$ for $h(.)$ a function that is continuous at $\boldsymbol{\beta}$
e.) $(\mathbf{R}\mathbf{b}-\mathbf{r})'[s^2\mathbf{R}(\mathbf{X}'\mathbf{X})^{-1}\mathbf{R}']^{-1}(\mathbf{R}\mathbf{b}-\mathbf{r}) \xrightarrow{L} \chi^2_m$ when $\mathbf{R}\boldsymbol{\beta} = \mathbf{r}$ for $\mathbf{R}$ an $(m \times k)$ matrix of rank $m \leq k$

2.) (105 points total) A researcher is interested in a relation between a scalar y_t and another scalar x_t^* that takes the form:

$$y_t = \beta x_t^* + \varepsilon_t.$$

The problem is that the researcher does not have observations on x_t^*, but does have observations on two other variables x_{1t} and x_{2t} which are related to x_t^* according to

$$x_{1t} = x_t^* + v_{1t}$$

$$x_{2t} = x_t^* + v_{2t}.$$

The magnitues v_{1t}, v_{2t}, x_t^*, and ε_t are independent i.i.d. Normal variables:

$$\begin{bmatrix} v_{1t} \\ v_{2t} \\ x_t^* \\ \varepsilon_t \end{bmatrix} \sim N\left(\begin{bmatrix} 0 \\ 0 \\ 0 \\ 0 \end{bmatrix}, \begin{bmatrix} \lambda_1^2 & 0 & 0 & 0 \\ 0 & \lambda_2^2 & 0 & 0 \\ 0 & 0 & \omega^2 & 0 \\ 0 & 0 & 0 & \sigma^2 \end{bmatrix} \right).$$

Thus the researcher has an observed sample of data on (y_t, x_{1t}, x_{2t}) for $t = 1, 2, ..., T$.

a.) (40 points) Consider the estimate $\hat{\alpha}$ from an OLS regression of y_t on x_{1t}:

$$y_t = \alpha x_{1t} + u_t.$$

Calculate the plim of $\hat{\alpha}_{OLS}$. Did you need any assumptions in addition to those stated above in order to calculate this plim? For what values of $(\lambda_1^2, \lambda_2^2, \omega^2, \sigma^2)$ would $\hat{\alpha}_{OLS}$ be a consistent estimate of β?

b.) (30 points) Suppose instead you estimated the value of α using an instrumental variables regression in which x_{2t} is used as an instrument for x_{1t}. Calculate the plim of $\hat{\alpha}_{IV}$. For what values of $(\lambda_1^2, \lambda_2^2, \omega^2, \sigma^2)$ would $\hat{\alpha}_{IV}$ be a consistent estimate of β?

c.) (10 points) For what values of $(\lambda_1^2, \lambda_2^2, \omega^2, \sigma^2)$ would you say that x_{2t} is a strong instrument for x_{1t}?

d.) (25 points) Suppose that you are worried that x_{2t} is a weak instrument for x_{1t}. Give the explicit formula and description for a test you could use to test the null hypothesis $H_0 : \beta = 2$ that would have exact small-sample size of 5% even if the instrument is weak.

3.) (95 points total) If $E[\mathbf{h}(\boldsymbol{\theta}_0, \mathbf{w}_t)] = \mathbf{0}$, the GMM estimate of $\boldsymbol{\theta}$ is defined as the value that minimizes

$$T[\mathbf{g}(\boldsymbol{\theta}; \mathbf{Y}_T)]'\mathbf{S}^{-1}[\mathbf{g}(\boldsymbol{\theta}; \mathbf{Y}_T)]$$

where

$$\mathbf{g}(\boldsymbol{\theta}; \mathbf{Y}_T) = T^{-1}\sum_{t=1}^{T}\mathbf{h}(\boldsymbol{\theta}, \mathbf{w}_t)$$

$$\mathbf{S} = \lim_{T\to\infty} T^{-1}\sum_{t=1}^{T}\sum_{v=-\infty}^{\infty} E\left\{[\mathbf{h}(\boldsymbol{\theta}_0, \mathbf{w}_t)]\,[\mathbf{h}(\boldsymbol{\theta}_0, \mathbf{w}_{t-v})]'\right\}.$$

The usual asymptotic distribution of the GMM estimator is

$$\sqrt{T}(\hat{\boldsymbol{\theta}}_T - \boldsymbol{\theta}_0) \xrightarrow{L} N(\mathbf{0}, \mathbf{V})$$

where

$$\mathbf{V} = (\mathbf{D}\mathbf{S}^{-1}\mathbf{D}')^{-1}$$

$$\mathbf{D}' = E\left\{ \left.\frac{\partial \mathbf{h}(\boldsymbol{\theta}; \mathbf{w}_t)}{\partial \boldsymbol{\theta}'}\right|_{\boldsymbol{\theta}=\boldsymbol{\theta}_0} \right\}.$$

In this question you are asked to apply these results to maximum likelihood estimation. A positive-valued random variable y_t is said to have a Rayleigh distribution with parameter λ if its density is given by

$$\begin{aligned} f(y_t) &= \frac{y}{\lambda}\exp\left(-\frac{y^2}{2\lambda}\right) && \text{for } y \geq 0 \\ &= 0 && \text{otherwise.} \end{aligned}$$

Some moments for the Rayleigh distribution are as follows:

$$E(y_t) = \sqrt{\lambda\pi/2}$$

$$E(y_t^2) = 2\lambda$$

$$\text{Var}(y_t) = 2\lambda - \frac{\lambda\pi}{2}$$

$$E(y_t^4) = 8\lambda^2.$$

Assume that we have an i.i.d. sample of size T, denoted $y_1, y_2, ..., y_T$, drawn from this distribution.

a.) (20 points) Calculate the log likelihood function and find the maximum likelihood estimate of λ.

b.) (20 points) Calculate the score associated with observation t and find its expectation.

c.) (35 points) Using your results from part (b), if you were to interpret the MLE of λ that you calculated in part (a) as an example of a generalized method of moments estimator, what value would correspond to each of the following magnitudes?

i.) $\mathbf{h}(\boldsymbol{\theta}, \mathbf{w}_t)$

ii.) $\mathbf{S}$

iii.) $\mathbf{D}$

d.) (10 points) Use the results from (c) and the general formula for the asymptotic variance of $\hat{\boldsymbol{\theta}}_{GMM}$ to find the asymptotic distribution of $\hat{\lambda}_{MLE}$.

e.) (10 points) Can you suggest an alternative way to calculate the asymptotic distribution of $\hat{\lambda}_{MLE}$ using only the formula you derived in part (a) and the Central Limit Theorem for the sample mean of an i.i.d. random variable?

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