Econ 220B Final Exam
Winter 2012

DIRECTIONS: No books or notes of any kind are allowed. Answer all questions on separate paper. 250 points are possible on this exam.

1.) (80 points total) Consider the following regression model:

$$\mathbf{y} = \mathbf{X}\boldsymbol{\beta} + \boldsymbol{\varepsilon}$$

$$E(\boldsymbol{\varepsilon}|\mathbf{X}) = \mathbf{0}$$

$$E(\boldsymbol{\varepsilon}\boldsymbol{\varepsilon}'|\mathbf{X}) = \sigma^2\mathbf{V}.$$

Here $\mathbf{X}$ is a $(T \times k)$ matrix of rank k and $\mathbf{V}$ is a known positive definite $(T \times T)$ matrix.

a.) (10 points) Write down the expression for $\mathbf{b}$, the OLS estimate of $\boldsymbol{\beta}$.

b.) (20 points) What would it mean to say that the estimate $\mathbf{b}$ is unbiased? Is it unbiased under the assumptions stated? (Note: you do not need to derive the result, just state whether or not it is unbiased).

c.) (10 points) Write down the expression for $\hat{\boldsymbol{\beta}}_{GLS}$, the GLS estimate of $\boldsymbol{\beta}$.

d.) (10 points) Which of the following two expressions is smaller? (Note: you do not need to derive the result, just state which is smaller).

$$\sum_{t=1}^{T}(y_t - \mathbf{x}_t'\mathbf{b})^2 \quad \text{versus} \quad \sum_{t=1}^{T}(y_t - \mathbf{x}_t'\hat{\boldsymbol{\beta}}_{GLS})^2$$

e.) (20 points) Write down the expression for the OLS F test of the null hypothesis that $\boldsymbol{\beta} = \mathbf{0}$.

f.) (10 points) Would the statistic you proposed in part (e) have an exact $F(k, T-k)$ distribution under the assumptions stated? If not, what additional assumptions would you need to make in order to conclude that the statistic you proposed in part (e) would have an exact $F(k, T-k)$ distribution?

2.) (80 points) Consider the following regression model,

$$y_t = z_t\beta + u_t$$

where z_t and u_t are stationary ergodic scalars with mean zero with variances $E(z_t^2) = \sigma_{zz}$, $E(u_t^2) = \sigma_{uu}$, and $E(z_t u_t) = \sigma_{zu} > 0$.

a.) (10 points) Based on the above assumptions, would you expect the OLS estimate of β to be unbiased? If not, which direction would you expect the bias to go?

b.) (20 points) Write down the expression for the OLS estimate of β and calculate its plim.

c.) (10 points) Suppose you had another stationary ergodic scalar x_t. What conditions would x_t need to satisfy in order for it to serve as a relevant and valid instrument?

d.) (20 points) Assuming the conditions you specified in part (c) are satisfied, write down the expression for the 2SLS estimate of β and calculate its plim.

e.) (20 points) Suppose you are persuaded that x_t is a valid instrument, but you are concerned that it may be weak. Describe how you would implement an Anderson-Rubin test of the null hypothesis that $\beta = 0$. State conditions under which the statistic you proposed has an exact Student t distribution, and state its degrees of freedom.

3.) (90 points total) If $E[\mathbf{h}(\boldsymbol{\theta}, \mathbf{w}_t)] = \mathbf{0}$, the GMM estimate of $\boldsymbol{\theta}$ is defined as the value that minimizes

$$T[\mathbf{g}(\boldsymbol{\theta}; \mathbf{Y}_T)]' \hat{\mathbf{S}}^{-1} [\mathbf{g}(\boldsymbol{\theta}; \mathbf{Y}_T)]$$

where

$$\mathbf{g}(\boldsymbol{\theta}; \mathbf{Y}_T) = T^{-1} \sum_{t=1}^{T} \mathbf{h}(\boldsymbol{\theta}, \mathbf{w}_t)$$

and $\hat{\mathbf{S}}$ is a consistent estimate of

$$\mathbf{S} = \lim_{T \to \infty} T^{-1} \sum_{t=1}^{T} \sum_{v=-\infty}^{\infty} E\left\{[\mathbf{h}(\boldsymbol{\theta}_0, \mathbf{w}_t)][\mathbf{h}(\boldsymbol{\theta}_0, \mathbf{w}_{t-v})]'\right\}.$$

The usual asymptotic distribution of the GMM estimator is

$$\sqrt{T}\,(\hat{\boldsymbol{\theta}}_T - \boldsymbol{\theta}_0) \xrightarrow{L} N(\mathbf{0}, \mathbf{V})$$

where $\mathbf{V}$ can be estimated consistently from

$$\hat{\mathbf{V}} = (\hat{\mathbf{D}} \hat{\mathbf{S}}^{-1} \hat{\mathbf{D}}')^{-1}$$

$$\hat{\mathbf{D}}' = \left. \frac{\partial \mathbf{g}(\boldsymbol{\theta}; \mathbf{Y}_T)}{\partial \boldsymbol{\theta}'} \right|_{\boldsymbol{\theta} = \hat{\boldsymbol{\theta}}}.$$

A random variable X is said to have a Pareto distribution if its density is given by

$$f_X(x) = \begin{cases} \alpha x^{-(1+\alpha)} & \text{if } x \geq 1 \\ 0 & \text{if } x < 1 \end{cases}.$$

Thus an i.i.d. sample of size T from this distribution would have log likelihood given by

$$\mathcal{L}(\mathbf{Y}_T, \alpha) = T \log \alpha - (1 + \alpha) \sum_{t=1}^{T} \log x_t.$$

For purposes of answering this question you can assume that $\alpha > 2,$ in which case some properties of the Pareto distribution that you may find helpful include the following:

$$E(X) = \frac{\alpha}{\alpha - 1}$$

$$\text{Var}(X) = \frac{\alpha}{(\alpha - 1)^2 (\alpha - 2)}$$

$$E[\log(X)] = 1/\alpha$$

$$\text{Var}[\log(X)] = 1/\alpha^2.$$

a.) (10 points) Find the first-order conditions associated with the maximum likelihood estimate $\hat{\alpha}_{MLE}$.

b.) (10 points) Interpret your answer to part (a) as a GMM estimator that sets $T^{-1}\sum_{t=1}^{T}\mathbf{h}(\boldsymbol{\theta},\mathbf{w}_t) = \mathbf{0}$. What corresponds to $\boldsymbol{\theta}$, $\mathbf{w}_t$, and $\mathbf{h}(\boldsymbol{\theta},\mathbf{w}_t)$ for this example?

c.) (20 points) Calculate the values of $\hat{\mathbf{D}}$ and $\hat{\mathbf{S}}$ associated with your answer to part (b), and use these to find the asymptotic distribution of $\hat{\alpha}_{MLE}$.

d.) (10 points) An alternative to maximum likelihood estimation is to use a GMM estimate based directly on the moment function

$$\mathbf{h}(\boldsymbol{\theta},\mathbf{w}_t) = x_t - \alpha/(\alpha - 1).$$

Find a closed-form expression for a GMM estimator $\hat{\alpha}_{GMM}$ using this moment condition.

e.) (20 points) Calculate the values of $\hat{\mathbf{D}}$ and $\hat{\mathbf{S}}$ associated with your answer to part (d), and use these to find the asymptotic distribution of $\hat{\alpha}_{GMM}$.

f.) (10 points) Prove that the asymptotic variance of $\hat{\alpha}_{GMM}$ is greater than the asymptotic variance of $\hat{\alpha}_{MLE}$. Hint: use the fact that

$$\frac{(\alpha-1)^2}{\alpha(\alpha-2)} = \frac{\alpha^2 - 2\alpha + 1}{\alpha^2 - 2\alpha} > 1.$$

g.) (10 points) Does your result in part (f) mean that something about the regularity conditions usually assumed for MLE and GMM fail to hold for this example? Explain.