Econ 220B Final Exam
Winter 2010

DIRECTIONS: No books or notes of any kind are allowed. Answer all questions on separate paper. 250 points are possible on this exam.

1.) (20 points total) A regression of a scalar y_t on a vector $\mathbf{x}_t$ produces an OLS regression coefficient given by

$$\mathbf{b} = \left[\sum_{t=1}^{T} \mathbf{x}_t \mathbf{x}_t'\right]^{-1} \left[\sum_{t=1}^{T} \mathbf{x}_t y_t\right].$$

Suppose that the vector $(y_t, \mathbf{x}_t')'$ is stationary and ergodic with

$$E\begin{bmatrix} y_t \\ \mathbf{x}_t \end{bmatrix} = \begin{bmatrix} \mu_y \\ \boldsymbol{\mu}_x \end{bmatrix}$$

$$E\begin{bmatrix} y_t^2 & y_t \mathbf{x}_t' \\ \mathbf{x}_t y_t & \mathbf{x}_t \mathbf{x}_t' \end{bmatrix} = \begin{bmatrix} \Sigma_{yy} & \boldsymbol{\Sigma}_{xy}' \\ \boldsymbol{\Sigma}_{xy} & \boldsymbol{\Sigma}_{xx} \end{bmatrix}.$$

a.) (10 points) Find the plim of $\mathbf{b}$.
b.) (10 points) Can you describe two different kinds of situations in which this plim of $\mathbf{b}$ would be the object that an economist would want to know?

2.) (90 points total) An economist has data on the following observations:
y_t = log of tons of steel produced in the U.S. in year t
ℓ_t = log of number of workers employed by U.S. steel industry in year t
k_t = log of real value of capital stock of U.S. steel industry in year t
n_t = log of energy consumed by steel industry in year t
The economist is interested in estimating a Cobb-Douglas production function of the form

$$y_t = \alpha_1 \ell_t + \alpha_2 k_t + \alpha_3 n_t + \varepsilon_t$$

where ε_t represents a shock to the production function. Let

$$\mathbf{y} = \begin{bmatrix} y_1 \\ \vdots \\ y_T \end{bmatrix} \quad \mathbf{X} = \begin{bmatrix} \ell_1 & k_1 & n_1 \\ \vdots & \vdots & \vdots \\ \ell_T & k_T & n_T \end{bmatrix} \quad \boldsymbol{\varepsilon} = \begin{bmatrix} \varepsilon_1 \\ \vdots \\ \varepsilon_T \end{bmatrix}$$

a.) (30 points) Suppose the economist believes that $\boldsymbol{\varepsilon}|\mathbf{X} \sim N(\mathbf{0}, \sigma^2\mathbf{I}_T)$ and wants to test the null hypothesis of constant returns to scale ($H_0 : \alpha_1 + \alpha_2 + \alpha_3 = 1$) against the one-sided alternative of decreasing returns to scale ($H_A : \alpha_1 + \alpha_2 + \alpha_3 < 1$). Suggest a test statistic that could be used to test this hypothesis, and explain exactly how this statistic would be used (e.g, "reject H_0 if $A > B$", where you tell me exactly what A and B are or where you could look up their values). Is this an exact small-sample test under the specified assumptions, or only an approximation? Note you only need to state the form of the statistic, and do not need to derive or prove anything.

b.) (30 points) Suppose instead that the economist believes that $\boldsymbol{\varepsilon}|\mathbf{X} \sim N(\mathbf{0}, \mathbf{V})$, where the economist believes that $\mathbf{V}$ is a diagonal matrix,

$$\mathbf{V} = \begin{bmatrix} v_1 & 0 & \cdots & 0 \\ 0 & v_2 & \cdots & 0 \\ \vdots & \vdots & \cdots & \vdots \\ 0 & 0 & \cdots & v_T \end{bmatrix}$$

but the economist does not know the particular values of $v_1, v_2, ..., v_T$. Suppose now that the economist wishes to test the null hypothesis ($H_0 : \alpha_1 + \alpha_2 + \alpha_3 = 1$) against the two-sided alternative ($H_A : \alpha_1 + \alpha_2 + \alpha_3 \neq 1$). Suggest a test statistic that could be used to test this hypothesis, and explain exactly how this statistic would be used. Is this an exact small-sample test under the specified assumptions, or only an approximation? Note you only need to state the form of the statistic, and do not need to derive or prove anything.

c.) (30 points) Do you think that the assumption that $\boldsymbol{\varepsilon}|\mathbf{X} \sim N(\mathbf{0}, \mathbf{V})$ is a reasonable one? Why or why not? Can you suggest any test the economist might use to test whether the concerns you raise are something to worry about for this example?

3.) (140 points total) If $E[\mathbf{h}(\boldsymbol{\theta}_0, \mathbf{w}_t)] = \mathbf{0}$, the GMM estimate of $\boldsymbol{\theta}$ is defined as the value that minimizes

$$T[\mathbf{g}(\boldsymbol{\theta}, \mathbf{Y}_T)]'\mathbf{S}^{-1}[\mathbf{g}(\boldsymbol{\theta}, \mathbf{Y}_T)]$$

where

$$\mathbf{g}(\boldsymbol{\theta}, \mathbf{Y}_T) = T^{-1}\sum_{t=1}^{T}\mathbf{h}(\boldsymbol{\theta}, \mathbf{w}_t)$$

$$\mathbf{S} = \lim_{T\to\infty} T^{-1}\sum_{t=1}^{T}\sum_{v=-\infty}^{\infty} E\left\{[\mathbf{h}(\boldsymbol{\theta}_0, \mathbf{w}_t)]\left[\mathbf{h}(\boldsymbol{\theta}_0, \mathbf{w}_{t-v})\right]'\right\}.$$

The first-order conditions for this minimization are

$$\left[\frac{\partial \mathbf{g}(\boldsymbol{\theta}, \mathbf{Y}_T)}{\partial \boldsymbol{\theta}'}\right]' \mathbf{S}^{-1}\mathbf{g}(\boldsymbol{\theta}, \mathbf{Y}_T) = \mathbf{0}.$$

The usual asymptotic distribution of the GMM estimator is

$$\sqrt{T}(\hat{\boldsymbol{\theta}}_T - \boldsymbol{\theta}_0) \xrightarrow{L} N(\mathbf{0}, \mathbf{V})$$

where

$$\mathbf{V} = (\mathbf{D}\mathbf{S}^{-1}\mathbf{D}')^{-1}$$

$$\mathbf{D}' = E\left\{\left.\frac{\partial \mathbf{h}(\boldsymbol{\theta}, \mathbf{w}_t)}{\partial \boldsymbol{\theta}'}\right|_{\boldsymbol{\theta}=\boldsymbol{\theta}_0}\right\}.$$

Hansen's J-statistic is given by

$$J = T[\mathbf{g}(\hat{\boldsymbol{\theta}}, \mathbf{Y}_T)]'\hat{\mathbf{S}}^{-1}[\mathbf{g}(\hat{\boldsymbol{\theta}}, \mathbf{Y}_T)]$$

for $\hat{\mathbf{S}}$ a consistent estimate of $\mathbf{S}$.

In this question you are asked to apply these results to the two-stage least squares regression model, in which our task is to estimate the $(a \times 1)$ vector $\boldsymbol{\theta}$ where

$$y_t = \mathbf{z}_t'\boldsymbol{\theta} + u_t$$

using an $(r \times 1)$ vector of instruments $\mathbf{x}_t$ satisfying $E(\mathbf{x}_t u_t) = \mathbf{0}$. You can assume that $r > a$, that the vector $(y_t, \mathbf{z}_t', \mathbf{x}_t')$ is stationary and ergodic, that $u_t\mathbf{x}_t$ is a martingale difference sequence, that $E(u_t^2|\mathbf{x}_t) = \sigma^2$, that the matrix $\boldsymbol{\Sigma}_{xx} = E(\mathbf{x}_t\mathbf{x}_t')$ has rank r, and that the matrix $\boldsymbol{\Sigma}_{xz} = E(\mathbf{x}_t\mathbf{z}_t')$ has rank a.

a.) (10 points) Are there any other assumptions besides those stated in the above paragraph that you would need to be able to apply the GMM results stated on page 3 to this particular example?

b.) (30 points) Find the values for $\mathbf{g}(\boldsymbol{\theta}, \mathbf{Y}_T)$, $\mathbf{S}$, and $\mathbf{D}'$ for this example.

c.) (20 points) Propose consistent estimates of $\mathbf{S}$ and $\mathbf{D}$, denoted $\hat{\mathbf{S}}$ and $\hat{\mathbf{D}}$, respectively.

d.) (20 points) Show that for this example the GMM estimate $\hat{\boldsymbol{\theta}}_{GMM}$ is equivalent to the 2SLS estimate

$$\hat{\boldsymbol{\theta}}_{2SLS} = \left[(\Sigma\mathbf{z}_t\mathbf{x}_t')\left(\Sigma\mathbf{x}_t\mathbf{x}_t'\right)^{-1}(\Sigma\mathbf{x}_t\mathbf{z}_t')\right]^{-1}\left[(\Sigma\mathbf{z}_t\mathbf{x}_t')\left(\Sigma\mathbf{x}_t\mathbf{x}_t'\right)^{-1}(\Sigma\mathbf{x}_t y_t)\right].$$

e.) (30 points) Show that Hansen's J statistic in this case is the same as Sargan's statistic,

$$\frac{\left(\Sigma\hat{u}_t\mathbf{x}_t'\right)\left(\Sigma\mathbf{x}_t\mathbf{x}_t'\right)^{-1}\left(\Sigma\mathbf{x}_t\hat{u}_t\right)}{\hat{\sigma}^2}$$

for $\hat{u}_t = y_t - \mathbf{z}_t'\hat{\boldsymbol{\theta}}$ and $\hat{\sigma}^2 = T^{-1}\Sigma\hat{u}_t^2$. What would be the asymptotic distribution you would assume for this statistic? Note: you do not have to derive this result, just state the distribution you would use.

f.) (30 points) Propose an adaptation of Sargan's statistic or Hansen's J statistic that might be used to test the null hypothesis $H_0 : \boldsymbol{\theta} = \boldsymbol{\theta}_0$ that does not require the matrix $\boldsymbol{\Sigma}_{xz}$ to have rank a. What would be the asymptotic distribution you would assume for this statistic? Prove that the statistic you have proposed does indeed have the distribution you claimed even when the rank of $\boldsymbol{\Sigma}_{xz}$ is less than a.