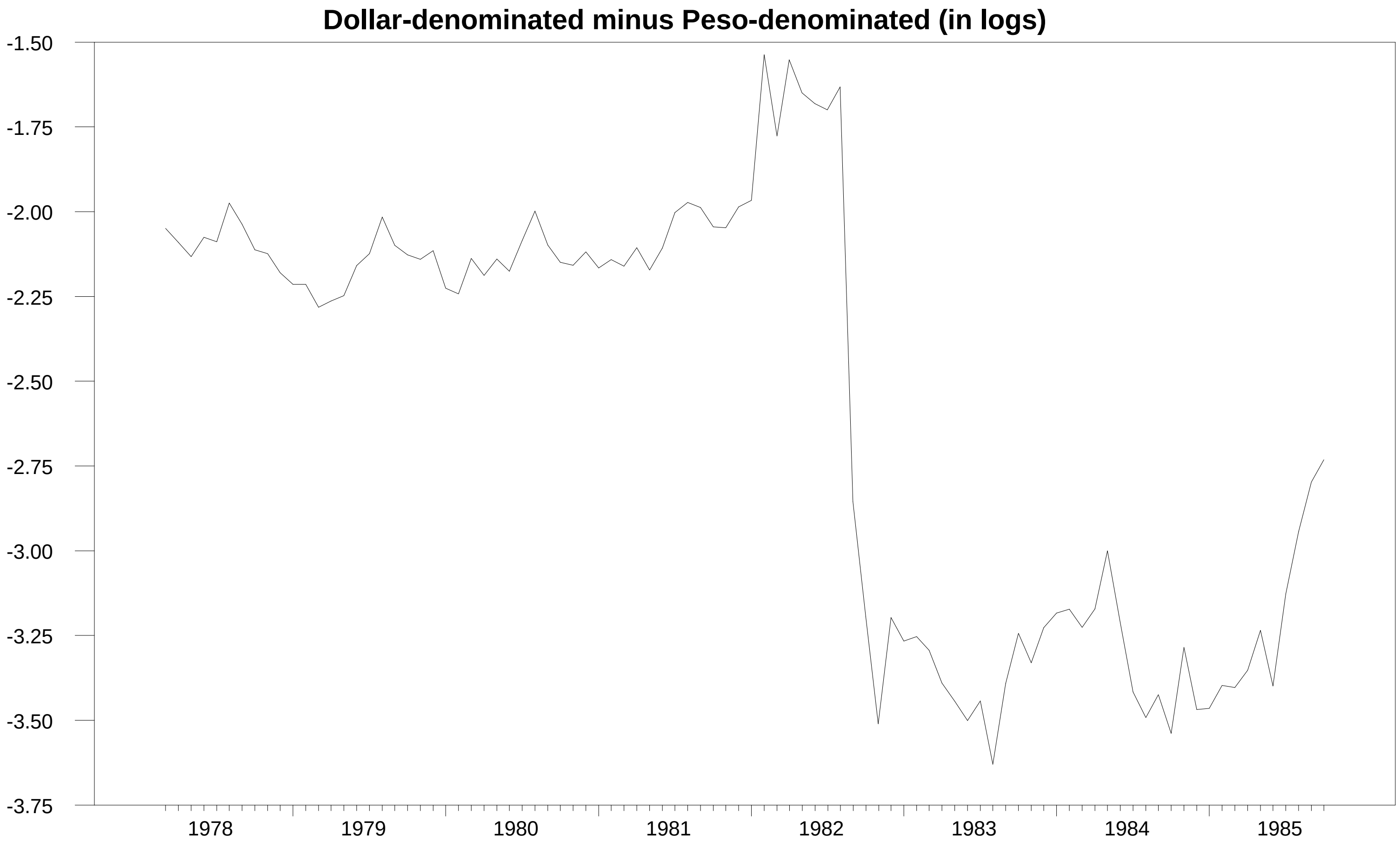
Business cycles and changes in regime

1. Motivating examples
2. Econometric approaches
3. Incorporating into theoretical models

1. Motivating examples

- Many economic series exhibit dramatic breaks:
  - recessions
  - financial panics
  - currency crises
- Questions:
  - how handle econometrically?
  - how incorporate into economic theory?

An example of change in regime



Model of structural change:

$$y_t - \mu_1 = \phi(y_{t-1} - \mu_1) + \varepsilon_t \qquad t \leq t_0$$

$$y_t - \mu_2 = \phi(y_{t-1} - \mu_2) + \varepsilon_t \qquad t > t_0$$

Questions:

1) How forecast with this model?

2) What caused change at t_0?

3) What is probability law for $\{y_t\}$?

$$s_t^* = 1 \quad t = 1, 2, .., t_0$$

$$s_t^* = 2 \quad t = t_0 + 1, t_0 + 2, \ldots$$

$$y_t - \mu_{s_t^*} = \phi(y_{t-1} - \mu_{s_{t-1}^*}) + \varepsilon_t$$

Need: probability law for s_t^*

Markov chain:

$$P(s_t^* = j | s_{t-1}^* = i, s_{t-2}^* = k, \ldots)$$
$$= P(s_t^* = j | s_{t-1}^* = i)$$
$$= p_{ij}$$

Transition from 1 to 2 is permanent

$$\Rightarrow p_{21} = 0$$

Economic recessions as changes in regime

y_t = real GDP growth in quarter t

$s_t = 1$ when economy is in expansion

$s_t = 2$ when economy is in recesion

$$y_t = m_{s_t} + \varepsilon_t$$

$$\varepsilon_t \sim N(0, \sigma^2)$$

$$\text{Prob}(s_t = j | s_{t-1} = i, s_{t-2} = k, \ldots, y_{t-1}, y_{t-2}, \ldots)$$
$$= p_{ij}$$

If s_t is observed, $m_{s_t} \sim$ AR(1)

$$m_{s_t} = a + \lambda m_{s_{t-1}} + v_t$$

$$a = p_{21} m_1 + p_{12} m_2$$

$$\lambda = p_{11} - p_{21}$$

$v_t \sim$ martingale difference sequence

$$E(m_{s_t} | s_{t-1} = 1) = p_{21} m_1 + p_{12} m_2 + (p_{11} - p_{21}) m_1$$

$$= p_{12} m_2 + p_{11} m_1 \quad \checkmark$$

$$E(m_{s_t} | s_{t-1} = 2) = p_{21} m_1 + p_{12} m_2 + (p_{11} - p_{21}) m_2$$

$$= p_{21} m_1 + (p_{12} + p_{11} - p_{21}) m_2$$

$$= p_{21} m_1 + (1 - p_{21}) m_2$$

$$= p_{21} m_1 + p_{22} m_2 \quad \checkmark$$

If only $\Omega_t = \{y_t, y_{t-1}, \ldots, y_1\}$ is observed, Prob$(s_t = 1|\Omega_t)$ is nonlinear in Ω_t.

Given Prob$(s_{t-1} = j|\Omega_{t-1})$, can calculate Prob$(s_t = j|\Omega_t)$ (and likelihood $f(y_t|\Omega_{t-1})$) recursively:

$$\text{Prob}(s_t = j|\Omega_{t-1}) = p_{1j}\text{Prob}(s_{t-1} = 1|\Omega_{t-1}) + p_{2j}\text{Prob}(s_{t-1} = 2|\Omega_{t-1})$$

$$f(y_t|s_t = i, \Omega_{t-1}) = \frac{1}{\sqrt{2\pi\sigma^2}} \exp\left(-\frac{(y_t - m_i)^2}{2\sigma^2}\right)$$

$$f(y_t|\Omega_{t-1}) = \sum_{i=1}^{2} \text{Prob}(s_t = i|\Omega_{t-1}) f(y_t|s_t = i, \Omega_{t-1})$$

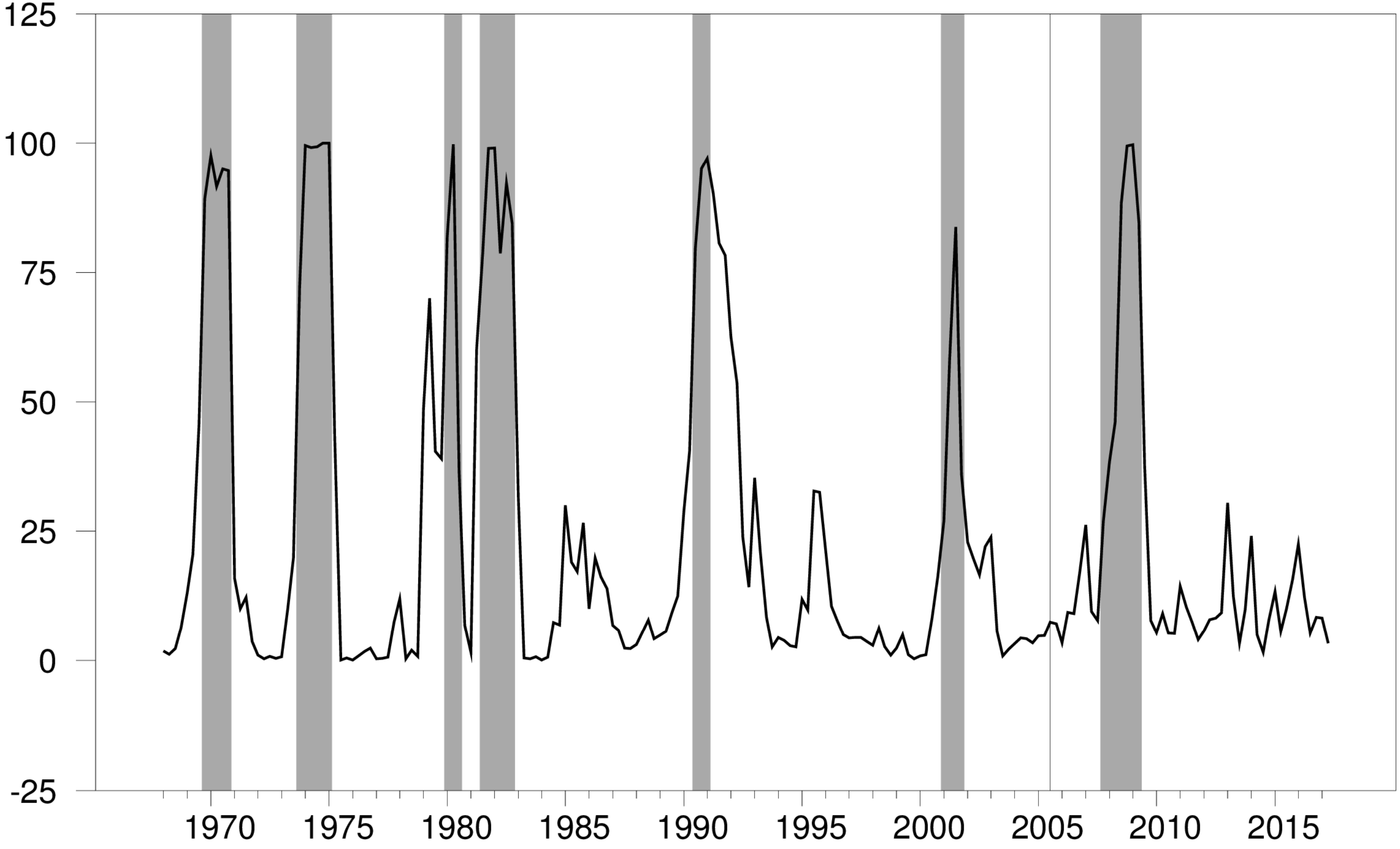
$$\text{Prob}(s_t = j|\Omega_t) = \frac{\text{Prob}(s_t = j|\Omega_{t-1}) f(y_t|s_t = j, \Omega_{t-1})}{f(y_t|\Omega_{t-1})}$$

Could choose population parameters $\boldsymbol{\theta} = (m_1, m_2, \sigma, p_{11}, p_{22})'$ by maximizing likelihood.

Plot of Prob($s_t = 2|\Omega_{t+1}, \hat{\boldsymbol{\theta}}_{t+1}$) with simulated real-time inference (historical real-time data sets from ALFRED) through 2005.

Plot of actual real-time inference (announced publicly at each date $t+1$) since 2005.

GDP-based Recession indicator index



Date of announcement	Announcement
Simulated (through June 2005)	
May 1970	recession began 1969:Q2
Aug 1971	recession ended 1970:Q4
May 1974	recession began 1973:Q4
Feb 1976	recession ended 1975:Q1
Nov 1979	recession began 1979:Q2
May 1981	recession ended 1980:Q2
Feb 1982	recession began 1981:Q2
Aug 1983	recession ended 1982:Q4
Feb 1991	recession began 1989:Q4
Feb 1993	recession ended 1991:Q4
Feb 2002	recession began 2001:Q1
Aug 2002	recession ended 2001:Q3
Actual real time (since July 2005)	
Jan 30, 2009	recession began 2007:Q4
Apr 30, 2010	recession ended 2009:Q2

2. Econometric treatment of changes in regime

2.1. Multivariate or non-Gaussian processes and multiple regimes

2.2. Time-varying transition probabilities

2.3. Processes that depend on current and past regimes

2.4. Latent-variable models with changes in regime

2.5. Analysis using Bayesian methods

2.6. Selecting the number of regimes

2.7. Deterministic breaks

2.8. Chib’s multiple change-point model

2.9. Smooth transition models

2.1. Multivariate or non-Gaussian processes and multiple regimes

Previous recursion used

$$f(y_t|\Omega_{t-1}) = \sum_{i=1}^{2} \text{Prob}(s_t = i|\Omega_{t-1}) f(y_t|s_t = i, \Omega_{t-1})$$

for $f(y_t|s_t = i)$ the $N(m_i, \sigma^2)$ density.

But works same for $f(\mathbf{y}_t|s_t = i, \Omega_{t-1})$

any multivariate, non-Gaussian density.

Krolzig (1997) switching VAR:

$$f(\mathbf{y}_t|\Omega_{t-1};\boldsymbol{\theta}_j) = \frac{1}{(2\pi)^{n/2}|\boldsymbol{\Sigma}_j|^{1/2}} \times$$

$$\exp\left[-(1/2)(\mathbf{y}_t - \boldsymbol{\mu}_{jt})'\boldsymbol{\Sigma}_j^{-1}(\mathbf{y}_t - \boldsymbol{\mu}_{jt})\right]$$

$$\boldsymbol{\mu}_{jt} = \mathbf{c}_j + \boldsymbol{\Phi}_{1j}\mathbf{y}_{t-1} + \boldsymbol{\Phi}_{2j}\mathbf{y}_{t-2} + \cdots + \boldsymbol{\Phi}_{rj}\mathbf{y}_{t-r}.$$

Dueker (1997): degrees of freedom on Student t change with regime.

In general, if s_t is a Markov chain taking on one of the values $s_t = 1, 2, \ldots, N$, let $p_{ij} = P(s_t = j | s_{t-1} = i)$. Collect in matrix $\mathbf{P} = [p_{ji}]$

$$\mathbf{P} = \begin{bmatrix} p_{11} & p_{21} & \cdots & p_{N1} \\ p_{12} & p_{22} & \cdots & p_{N2} \\ \vdots & \vdots & \cdots & \vdots \\ p_{1N} & p_{2N} & \cdots & p_{NN} \end{bmatrix}$$

Let $\boldsymbol{\xi}_t = \mathbf{e}_i$ (the ith column of $\mathbf{I}_N$) when $s_t = i$. Then

$$E(\boldsymbol{\xi}_{t+1}|\boldsymbol{\xi}_t = \mathbf{e}_i) = \begin{bmatrix} P(s_{t+1} = 1|s_t = i) \\ P(s_{t+1} = 2|s_t = i) \\ \vdots \\ P(s_{t+1} = N|s_t = i) \end{bmatrix}$$

$$= \mathbf{P}\mathbf{e}_i$$

$$= \mathbf{P}\boldsymbol{\xi}_t$$

$$\boldsymbol{\xi}_t = \mathbf{P}\boldsymbol{\xi}_{t-1} + \mathbf{v}_t$$

$\mathbf{v}_t \sim$ martingale difference sequence.

In other words, N-state Markov chain can be represented using VAR(1).

Suppose we had a set of observations $\Omega_t = \{\mathbf{y}_t, \mathbf{y}_{t-1}, \ldots, \mathbf{y}_1\}$ that gave us an imperfect inference about s_t summarized as

$$\hat{\boldsymbol{\xi}}_{t|t} = E(\boldsymbol{\xi}_t|\Omega_t) = \begin{bmatrix} P(s_t = 1|\Omega_t) \\ P(s_t = 2|\Omega_t) \\ \vdots \\ P(s_t = N|\Omega_t) \end{bmatrix}$$

Then

$$\widehat{\boldsymbol{\xi}}_{t+1|t} = E(\boldsymbol{\xi}_{t+1}|\Omega_t) = \mathbf{P}\widehat{\boldsymbol{\xi}}_{t|t}$$

(e.g., row j states that

$$P(s_{t+1} = j|\Omega_t)$$
$$= p_{1j}P(s_t = 1|\Omega_t) + p_{2j}P(s_t = 2|\Omega_t)$$
$$+\cdots + p_{Nj}P(s_t = N|\Omega_t)$$

Collect the densities that might be associated with each of the N states in an $(N \times 1)$ vector

$$\boldsymbol{\eta}_t = \begin{bmatrix} p(y_t | s_t = 1, \Omega_{t-1}) \\ p(y_t | s_t = 2, \Omega_{t-1}) \\ \vdots \\ p(y_t | s_t = N, \Omega_{t-1}) \end{bmatrix}$$

Recall that

$$\mathbf{P}\hat{\boldsymbol{\xi}}_{t-1|t-1} = \begin{bmatrix} P(s_t = 1|\Omega_{t-1}) \\ P(s_t = 2|\Omega_{t-1}) \\ \vdots \\ P(s_t = N|\Omega_{t-1}) \end{bmatrix}$$

Thus

$$\boldsymbol{\eta}_t \odot \mathbf{P}\widehat{\boldsymbol{\xi}}_{t-1|t-1} =$$

$$\begin{bmatrix} p(y_t|s_t = 1, \Omega_{t-1})P(s_t = 1|\Omega_{t-1}) \\ p(y_t|s_t = 2, \Omega_{t-1})P(s_t = 2|\Omega_{t-1}) \\ \vdots \\ p(y_t|s_t = N, \Omega_{t-1})P(s_t = N|\Omega_{t-1}) \end{bmatrix}$$

Summing the elements of this vector gives

$$\mathbf{1}'(\boldsymbol{\eta}_t \odot \mathbf{P}\hat{\boldsymbol{\xi}}_{t-1|t-1})$$

$$= \sum_{j=1}^{N} p(y_t, s_t = j|\Omega_{t-1})$$

$$= p(y_t|\Omega_{t-1}),$$

the conditional likelihood of *t*th observation.

The result of dividing the *j*th element of $(\boldsymbol{\eta}_t \odot \mathbf{P}\hat{\boldsymbol{\xi}}_{t-1|t-1})$ by the conditional likelihood is

$$\frac{p(y_t, s_t = j|\Omega_{t-1})}{p(y_t|\Omega_{t-1})} = P(s_t = j|y_t, \Omega_{t-1})$$

$$\frac{(\boldsymbol{\eta}_t \odot \mathbf{P}\hat{\boldsymbol{\xi}}_{t-1|t-1})}{\mathbf{1}'(\boldsymbol{\eta}_t \odot \mathbf{P}\hat{\boldsymbol{\xi}}_{t-1|t-1})} = \hat{\boldsymbol{\xi}}_{t|t}$$

$$\frac{(\boldsymbol{\eta}_t \odot \mathbf{P}\hat{\boldsymbol{\xi}}_{t-1|t-1})}{\mathbf{1}'(\boldsymbol{\eta}_t \odot \mathbf{P}\hat{\boldsymbol{\xi}}_{t-1|t-1})} = \hat{\boldsymbol{\xi}}_{t|t}$$

Iterative algorithm similar to Kalman filter:

Input for step t:

$$\hat{\boldsymbol{\xi}}_{t-1|t-1}$$

(an $N \times 1$ vector whose jth element is

$P(s_t = j|y_t, y_{t-1}, \ldots, y_1)$).

Output for step t:

$$\hat{\boldsymbol{\xi}}_{t|t}$$

Options for initial value $\hat{\boldsymbol{\xi}}_{0|0}$:

(1) If Markov chain is ergodic, use ergodic probabilities

$$\hat{\boldsymbol{\xi}}_{0|0} = (\mathbf{A}'\mathbf{A})^{-1}\mathbf{A}'\mathbf{e}_{N+1}$$

$$\underset{(N+1)\times N}{\mathbf{A}} = \begin{bmatrix} \mathbf{I}_N - \mathbf{P} \\ \mathbf{1}' \end{bmatrix}$$

(2) Set $\hat{\boldsymbol{\xi}}_{0|0} = \boldsymbol{\rho}$, a vector of free parameters to be estimated by maximum likelihood or Bayesian methods along with the other parameters.

(3) Set $\hat{\boldsymbol{\xi}}_{0|0} = N^{-1}\mathbf{1}$.

(4) Set $\hat{\boldsymbol{\xi}}_{0|0}$ based on prior beliefs.

Above assumed we knew parameters $\boldsymbol{\theta}$ appearing in $\boldsymbol{\eta}_t = [p(y_t|s_t = j, \Omega_{t-1}; \boldsymbol{\theta})]_{j=1}^N$ (in first example $\boldsymbol{\theta} = (\phi, \mu_1, \mu_2, \sigma^2)'$) and $\mathbf{p}$ appearing in $\mathbf{P}$ (in this case $\mathbf{p} = (p_{11}, p_{22})'$).

However, as byproduct of step t of iteration we ended up calculating $p(y_t|\Omega_{t-1};\boldsymbol{\theta},\mathbf{p})$ and so we've calculated log likelihood

$$\mathcal{L}(\boldsymbol{\theta},\mathbf{p}) = \sum_{t=1}^{T} \log p(y_t|\Omega_{t-1};\boldsymbol{\theta},\mathbf{p})$$

which can be maximized numerically with respect to $\boldsymbol{\theta}$ and $\mathbf{p}$ by numerical methods.

Note– during numerical search we'd want to be choosing λ_{11} and λ_{22} rather than p_{11} and p_{22} where

$$p_{11} = \frac{\lambda_{11}^2}{1+\lambda_{11}^2}$$

$$p_{22} = \frac{\lambda_{22}^2}{1+\lambda_{22}^2}$$

General case:

$$\boldsymbol{\eta}_t = \begin{bmatrix} p(\mathbf{y}_t | s_t = 1, \Omega_{t-1}) \\ p(\mathbf{y}_t | s_t = 2, \Omega_{t-1}) \\ \vdots \\ p(\mathbf{y}_t | s_t = N, \Omega_{t-1}) \end{bmatrix}$$

$$p(\mathbf{y}_t | \Omega_{t-1}) = \mathbf{1}'(\boldsymbol{\eta}_t \odot \mathbf{P}\hat{\boldsymbol{\xi}}_{t-1|t-1})$$

$$\mathcal{L}(\mathbf{y}_1, \ldots, \mathbf{y}_T; \boldsymbol{\theta}) = \sum_{t=1}^{T} \log p(\mathbf{y}_t | \Omega_{t-1})$$

2.2. Time-varying transition probabilities

Simple recursion used

$$\text{Prob}(s_t = j|\Omega_{t-1}) = p_{1j}\text{Prob}(s_{t-1} = 1|\Omega_{t-1})$$
$$+ p_{2j}\text{Prob}(s_{t-1} = 2|\Omega_{t-1}).$$

But works the same when p_{1j} is replaced by any known function $p_{1j}(\Omega_{t-1})$.

2.3. Processes that depend on current and past regimes

Simple example assumed density $f(\mathbf{y}_t | s_t, \Omega_{t-1})$ depends only on current regime s_t.

If instead depends on current and past regimes can simply stack regimes as in companion form for VAR(p).

$$s_t^* = \begin{cases} 1 & \text{when } s_t = 1 \text{ and } s_{t-1} = 1 \\ 2 & \text{when } s_t = 2 \text{ and } s_{t-1} = 1 \\ 3 & \text{when } s_t = 1 \text{ and } s_{t-1} = 2 \\ 4 & \text{when } s_t = 2 \text{ and } s_{t-1} = 2 \end{cases}$$

$$\text{Prob}(s_t^* = j | s_{t-1}^* = i) = p_{ij}^* \quad i,j = 1, \ldots, 4.$$

2.4. Latent variable models with changes in regime

$$F_t = \alpha_{s_t} + \phi F_{t-1} + \eta_t$$

$$\mathbf{y}_t = \boldsymbol{\psi} F_t + \mathbf{q}_t$$

$$q_{jt} = \phi_j q_{j,t-1} + v_{jt}$$

- Can approximate likelihood function and optimal inference
    - Kim (1994)
- Useful for real-time inference
    - Chauvet and Hamilton (2006)
    - Chauvet and Piger (2008)
    - Camacho and Perez-Quiros (forthcoming)

2.5. Analysis using Bayesian methods

- Gibbs sampler
  - Albert and Chib (1993)
  - Kim and Nelson (1999)
- Time-varying transition probabilities
  - Filardo and Gordon (1998)
- Label-switching problem

  Frühwirth-Schnatter (2001)

2.6. Selecting the number of regimes

Smith, Naik and Tsai (2006):

$$y_t = \mathbf{x}_t' \boldsymbol{\beta}_{s_t} + \sigma_{s_t} \varepsilon_t$$

$$\hat{T}_i = \sum_{t=1}^{T} \text{Prob}(s_t = i | \Omega_T; \hat{\boldsymbol{\theta}}_{MLE})$$

$$MSC = -2\mathcal{L}(\hat{\boldsymbol{\theta}}_{MLE}) + \sum_{i=1}^{N} \frac{\hat{T}_i(\hat{T}_i + Nk)}{\hat{T}_i - Nk - 2}$$

- Calculate nonstandard properties of likelihood ratio test
    - Hansen (1992)
    - Garcia (1998)
- Use general specification tests of null of *N* regimes that have power against *N* +1
    - Hamilton (1996)
    - Carrasco, Hu and Ploberger (2014)

2.7. Deterministic breaks

- If breaks are deterministic, test as in

  Bai and Perron (1998, 2003)

- How forecast?

  - Pesaran and Timmermann (2007)

2.8. Chib's multiple change-point model

$$\mathbf{P} = \begin{bmatrix} p_{11} & 0 & 0 & \cdots & 0 & 0 \\ 1-p_{11} & p_{22} & 0 & \cdots & 0 & 0 \\ 0 & 1-p_{22} & p_{33} & \cdots & 0 & 0 \\ \vdots & \vdots & \vdots & \cdots & \vdots & \vdots \\ 0 & 0 & 0 & \cdots & p_{N-1,N-1} & 0 \\ 0 & 0 & 0 & \cdots & 1-p_{N-1,N-1} & 1 \end{bmatrix}$$

2.9. Smooth transition models

Teräsvirta (2004)

$$y_t = \frac{\exp[-\gamma(z_{t-1}-c)]}{1+\exp[-\gamma(z_{t-1}-c)]}\mathbf{x}'_{t-1}\boldsymbol{\beta}_1 + \frac{1}{1+\exp[-\gamma(z_{t-1}-c)]}\mathbf{x}'_{t-1}\boldsymbol{\beta}_2 + u_t$$

By contrast, in Markov-switching regression the switching weights Prob($s_{t-1} = i|\Omega_{t-1}$) depend on $y_{t-1}, y_{t-2}, \ldots, y_1$ not just z_{t-1}.

3. Economic theory and changes in regime

3.1. Closed-form solution of DSGE’s and asset-pricing implications

3.2. Approximating the solution to DSGE’s using perturbation methods

3.3. Linear rational expectations models with changes in regime

3.4. Multiple equilibria

3.5. Tipping points and financial crises

3.6. Currency crises and sovereign debt crises

3.7. Changes in policy as the source of changes in regime

3.1. Closed-form solution of DSGE's and asset-pricing implications

Lucas tree model with CRRA utility:

P_t = price of stock

D_t = dividend

γ = coefficient of relative risk aversion

$$P_t = D_t^{-\gamma} \sum_{k=1}^{\infty} \beta^k E_t D_{t+k}^{(1+\gamma)}$$

Cecchetti, Lam and Mark (1990):

$$\log D_t - \log D_{t-1} = m_{s_t} + \varepsilon_t$$

$$P_t = \rho_{s_t} D_t$$

- Portfolio allocation

  - Ang and Bekaert (2002a); Guidolin and Timmermann (2008)
- Financial implications of rare-event risk

  - Evans (1996); Barro (2006)
- Option pricing

  - Elliott, Chan and Siu (2005)
- Term structure of interest rates

  - Ang and Bekaert (2002b); Bansal and Zhou (2002)

3.2. Approximating the solution to DSGE's using perturbation methods

$$E_t \mathbf{a}(\mathbf{y}_{t+1}, \mathbf{y}_t, \mathbf{x}_t, \mathbf{x}_{t-1}, \boldsymbol{\varepsilon}_{t+1}, \boldsymbol{\varepsilon}_t, \boldsymbol{\theta}_{s_{t+1}}, \boldsymbol{\theta}_{s_t}) = \mathbf{0}$$

$\mathbf{y}_t$ = control variables

$\mathbf{x}_t$ = predetermined variables

$\boldsymbol{\varepsilon}_t$ = innovations to exogenous variables

s_t follows an N-state Markov chain

Cecchetti, Lam and Mark:

$$y_t = P_t / D_t$$

$$x_t = \ln(D_t / D_{t-1})$$

$$\theta_{s_t} = m_{s_t},$$

In general we seek solutions of the form

$$\mathbf{y}_t = \boldsymbol{\rho}_{s_t}(\mathbf{x}_{t-1}, \boldsymbol{\varepsilon}_t)$$

$$\mathbf{x}_t = \mathbf{h}_{s_t}(\mathbf{x}_{t-1}, \boldsymbol{\varepsilon}_t)$$

Foerster, et. al. (2014):

Sequence of economies indexed by χ

$\chi \to 0 \Rightarrow$ deterministic steady state

$\chi \to 1 \Rightarrow$ previous solution

$$\mathbf{y}_t = \boldsymbol{\rho}_{s_t}(\mathbf{x}_{t-1}, \boldsymbol{\varepsilon}_t, \chi)$$

$$\mathbf{x}_t = \mathbf{h}_{s_t}(\mathbf{x}_{t-1}, \boldsymbol{\varepsilon}_t, \chi)$$

$\mathbf{x}^*, \mathbf{y}^*$ are steady-state solution when $\chi = 0,\ \boldsymbol{\varepsilon}_t = \mathbf{0},\ \boldsymbol{\theta}_{s_t} = \boldsymbol{\theta}^* =$ ergodic value from Markov chain.

$$\mathbf{y}_t = \mathbf{y}^* + \mathbf{R}^{x}_{s_t}(\mathbf{x}_{t-1} - \mathbf{x}^*) + \mathbf{R}^{\varepsilon}_{s_t}\boldsymbol{\varepsilon}_t + \mathbf{R}^{\chi}_{s_t}$$

$$\mathbf{x}_t = \mathbf{x}^* + \mathbf{H}^{x}_{s_t}(\mathbf{x}_{t-1} - \mathbf{x}^*) + \mathbf{H}^{\varepsilon}_{s_t}\boldsymbol{\varepsilon}_t + \mathbf{H}^{\chi}_{s_t}$$

3.3. Linear rational expectations models with changes in regime

$$\mathbf{A}_{s_t} E(\mathbf{y}_{t+1} | \Omega_t, s_t, s_{t-1}, \ldots, s_1) = \mathbf{d}_{s_t} + \mathbf{B}_{s_t} \mathbf{y}_t + \mathbf{C}_{s_t} \mathbf{x}_t$$

$\mathbf{A}_j = (n_y \times n_y)$ matrix of parameters

when $s_t = j$.

Davig and Leeper (2007):

Let $\mathbf{y}_{jt}$ = value of $\mathbf{y}_t$ when $s_t = j$

$$\underset{(Nn_y \times 1)}{\mathbf{Y}_t} = \begin{bmatrix} \underset{(n_y \times 1)}{\mathbf{y}_{1t}} \\ \vdots \\ \underset{(n_y \times 1)}{\mathbf{y}_{Nt}} \end{bmatrix}$$

$$E(\mathbf{y}_{t+1}|s_t = i, \Omega_t) = \sum_{j=1}^{N} E(\mathbf{y}_{t+1}|s_{t+1} = j, s_t = i, \Omega_t) p_{ij}$$

Hence when $s_t = i$,

$$\mathbf{A}_{s_t} E(\mathbf{y}_{t+1}|s_t, \Omega_t) = (\mathbf{p}_i' \otimes \mathbf{A}_i) E(\mathbf{Y}_{t+1}|\mathbf{Y}_t)$$

$$\underset{(Nn_y \times Nn_y)}{\mathbf{A}} = \begin{bmatrix} \underset{(1\times N)}{\mathbf{p}_1'} \otimes \underset{(n_y \times n_y)}{\mathbf{A}_1} \\ \vdots \\ \underset{(1\times N)}{\mathbf{p}_N'} \otimes \underset{(n_y \times n_y)}{\mathbf{A}_N} \end{bmatrix} \qquad \underset{(Nn_y \times 1)}{\mathbf{d}} = \begin{bmatrix} \underset{(n_y \times 1)}{\mathbf{d}_1} \\ \vdots \\ \underset{(n_y \times 1)}{\mathbf{d}_N} \end{bmatrix}$$

$$\underset{(Nn_y \times Nn_y)}{\mathbf{B}} = \begin{bmatrix} \mathbf{B}_1 & \mathbf{0} & \cdots & \mathbf{0} \\ \mathbf{0} & \mathbf{B}_2 & \cdots & \mathbf{0} \\ \vdots & \vdots & \cdots & \vdots \\ \mathbf{0} & \mathbf{0} & \cdots & \mathbf{B}_N \end{bmatrix} \qquad \underset{(Nn_y \times n_x)}{\mathbf{C}} = \begin{bmatrix} \underset{(n_y \times n_x)}{\mathbf{C}_1} \\ \vdots \\ \underset{(n_y \times n_x)}{\mathbf{C}_N} \end{bmatrix}$$

Consider non-regime-changing system

$$\mathbf{A}E(\mathbf{Y}_{t+1}|\mathbf{Y}_t) = \mathbf{d} + \mathbf{B}\mathbf{Y}_t + \mathbf{C}\mathbf{x}_t$$

If we can find a stable solution of the form

$$\underset{(Nn_y\times 1)}{\mathbf{Y}_t} = \underset{(Nn_y\times 1)}{\mathbf{h}} + \underset{(Nn_y\times n_x)}{\mathbf{H}} \underset{(n_x\times 1)}{\mathbf{x}_t}$$

then the *i*th block

$$\underset{(n_y\times 1)}{\mathbf{y}_t} = \underset{(n_y\times 1)}{\mathbf{h}_{s_t}} + \underset{(n_y\times n_x)}{\mathbf{H}_{s_t}} \underset{(n_x\times 1)}{\mathbf{x}_t}$$

is a stable solution to our original equation of interest.

- However, even if we find a unique stable solution to the invariant system, there may be other stable solutions to the original system

  - Farmer, Waggoner, and Zha (2010)

3.4. Multiple equilibria

- Multiplicity of stable equilibria could itself be of interest

  - Coordination externalities (Cooper and John, 1988; Cooper, 1994)

  - Equilibria indexed by expectations (Kurz and Motolese, 2001)

- Regime-switching model could describe transitions between equilibria

  - Kirman (1993); Chamley (1999)

- Stock market bubbles

  - Hall, Psaradakis and Sola (1999)

- Difficult to distinguish from unobserved fundamentals

  - Hamilton (1985); Driffill and Sola (1998); Gürkaynak (2008)

3.5. Tipping points and financial crises

- In other models, there is a unique equilibrium, but small change in fundamentals can cause big change in outcome

 - Acemoglu and Scott (1997); Moore and Schaller (2002); Guo, Miao, and Morelle (2005); Veldkamp (2005); Startz (1998); Hong, Stein, and Yu (2007); Branch and Evans (2010)

- Financial crises

 - Brunnermeier and Sannikov (2014); Hamilton (2005); Asea and Blomberg (1998); Hubrich and Tetlow (2013)

3.6. Currency crises and sovereign debt crises

- Currency crises

  - Jeanne and Masson (2000); Peria (2002); Cerra and Saxena (2005)

- Sovereign debt crises

  - Greenlaw, et. al. (2013); Davig, Leeper and Walker (2011); Bi (2012)

3.7. Changes in policy as the source of changes in regime

- Monetary policy: hawks vs. doves

  Owyang and Ramey (2004); Schorfheide (2005); Liu, Waggoner, and Zha (2011); Bianchi (2013)

- Unsustainable fiscal policy and inflation

  Ruge-Murcia (1995, 1999)