- Class Monday Nov 13 will meet at 11:00 rather than 8:00.

Introduction to term structure of interest rates

A. Basic concepts

1. Prices, yields, and logs

Loan \$1 today, get \$$(1 + r)$ next year

$r =$ annual yield

$r = 0.05 \Rightarrow 5\%$ return

Compounded quarterly:

$(1 + r/4)^4$

Continuous compounding:

$$\lim_{m \to \infty} (1 + r/m)^m = e^r$$

e.g. $e^{0.05} = 1.051$

Continuous compounding

for n years:

$$e^r \bullet e^r \cdots e^r = e^{rn}$$

Let P_{nt} = price today for security that promises \$1 with certainty at $t+n$

$$P_{0t} \equiv 1$$

Called a pure discount bond

Continuously compounded return is characterized by value r for which

$$P_{nt} e^{nr} = 1$$

$$\log P_{nt} = -n r_{nt}$$

$$r_{nt} = -n^{-1} p_{nt}$$

for p_{nt} log of bond price and r_{nt} the annual yield to maturity on the bond

$$p_{nt} \uparrow \;\Rightarrow\; r_{nt} \downarrow$$

Introduction to term structure of interest rates

A. Basic concepts

1. Prices, yields, and logs
2. Holding period yield

My (continuously compounded) holding return is

$$p_{n-1,t+1} - p_{nt} = nr_{nt} - (n-1)r_{n-1,t+1}$$
$$= r_{nt} - (n-1)(r_{n-1,t+1} - r_{nt})$$

If rates rise ($r_{n-1,t+1} > r_{nt}$), then I earn less than r_{nt}.

Even though there is no default risk with the bond (it will be worth \$1 with certainty at $t+n$), there is interest-rate risk (I may earn more or less than r_{nt} if I hold for one year and then sell).

Introduction to term structure of interest rates

A. Basic concepts

1. Prices, yields, and logs
2. Holding period yield
3. Forward rate

Suppose I simultaneously sell \$1 in n-year bond (so I will have to pay $1/P_{nt}$ at $t+n$) and buy \$1 in $(n+1)$-year bond (so I will receive $1/P_{n+1,t}$ at $t+n+1$).

No cash flow between $t, t+1, \ldots, t+n-1$.

I can thus lock in today a return on a 1-period bond that I will purchase at $t+n$

$$f_{nt} = n\text{-year-ahead forward rate at } t$$

$$f_{nt} = p_{nt} - p_{n+1,t}$$

Introduction to term structure of interest rates

A. Basic concepts

1. Prices, yields, and logs
2. Holding period yield
3. Forward rate
4. Coupon bonds

Consider now a bond that will be worth \$1 at $t+n$ but also pays a coupon of \$$C$ at the end of $t, t+1, \ldots, t+n$. Can think of this as a set of n different pure-discount bonds whose price today should be $CP_{1t} + CP_{2t} + \cdots CP_{nt} + P_{nt}$.

Introduction to term structure of interest rates

A. Basic concepts

1. Prices, yields, and logs
2. Holding period yield
3. Forward rate
4. Coupon bonds
5. Yield curve

Gürkaynak, Sack and Wright (JME, 2007)

Suppose we conjecture that for given t, the forward rate f_{nt} is a smooth function of n:

$$
\begin{aligned}
f_{nt} = {} & \beta_{0t} + \beta_{1t} \exp(-n/\tau_{1t}) \\
& + \beta_{2t}(n/\tau_{1t}) \exp(-n/\tau_{1t}) \\
& + \beta_{3t}(n/\tau_{2t}) \exp(-n/\tau_{2t})
\end{aligned}
$$

Different values of $(\beta_{0t}, \beta_{1t}, \beta_{2t}, \beta_{3t}, \tau_{1t}, \tau_{2t})$ for each t

Recalling that $f_{nt} = p_{nt} - p_{n+1,t}$ and $p_{0t} = 0,$ we know

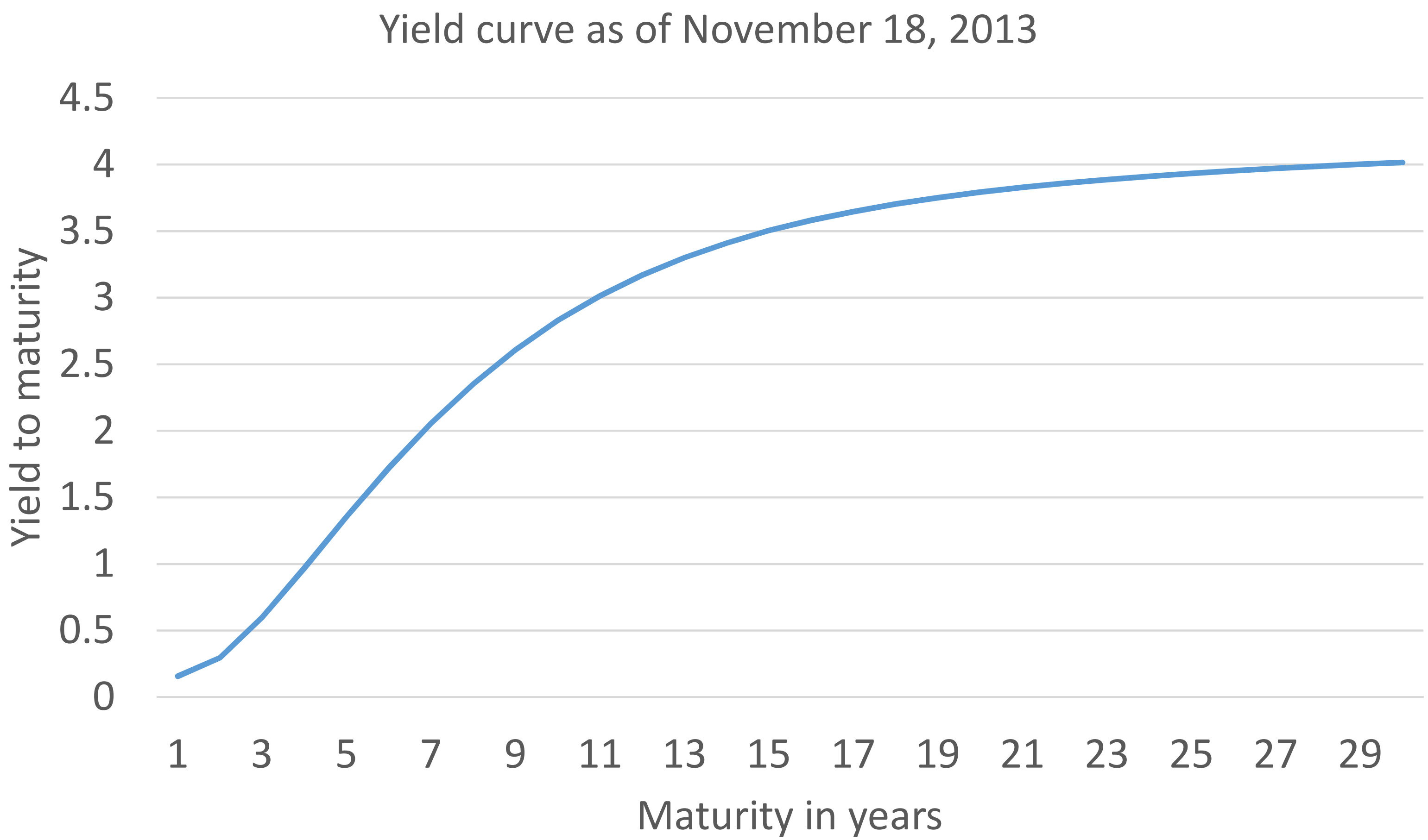
$$p_{n+1,t} = -f_{nt} - f_{n-1,t} - \cdots - f_{0t}$$

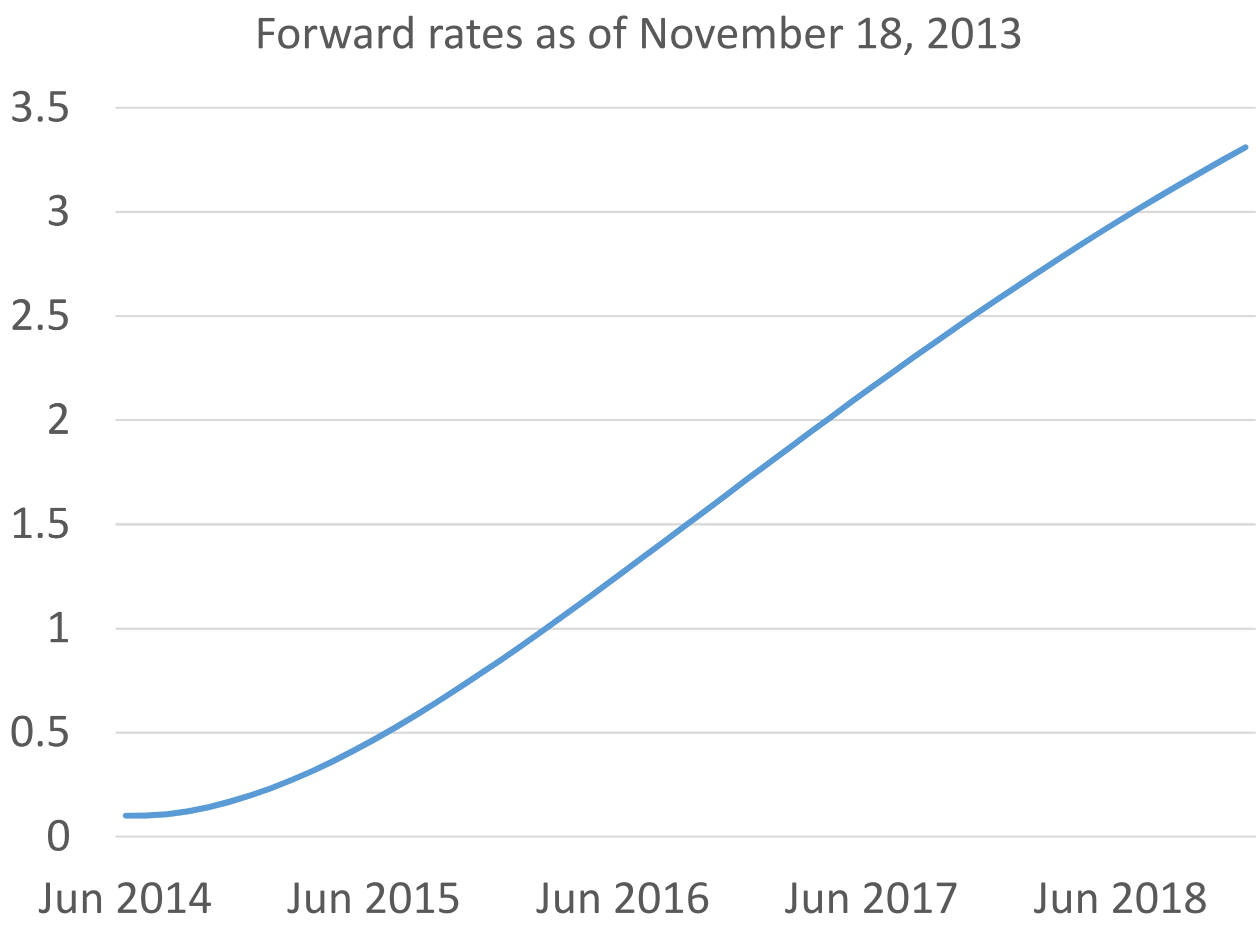
so given $(\beta_{0t}, \beta_{1t}, \beta_{2t}, \beta_{3t}, \tau_{1t}, \tau_{2t})$ we could calculate predicted price of any bond and choose $(\beta_{0t}, \beta_{1t}, \beta_{2t}, \beta_{3t}, \tau_{1t}, \tau_{2t})$ to best fit observed bond prices at t.

Actually, GSW use instantaneous forward rates (I lend \$1 for one day beginning n years from now) instead of the one-year forward rates (I lend \$1 for one year beginning n years from now), in which case above formula is instead

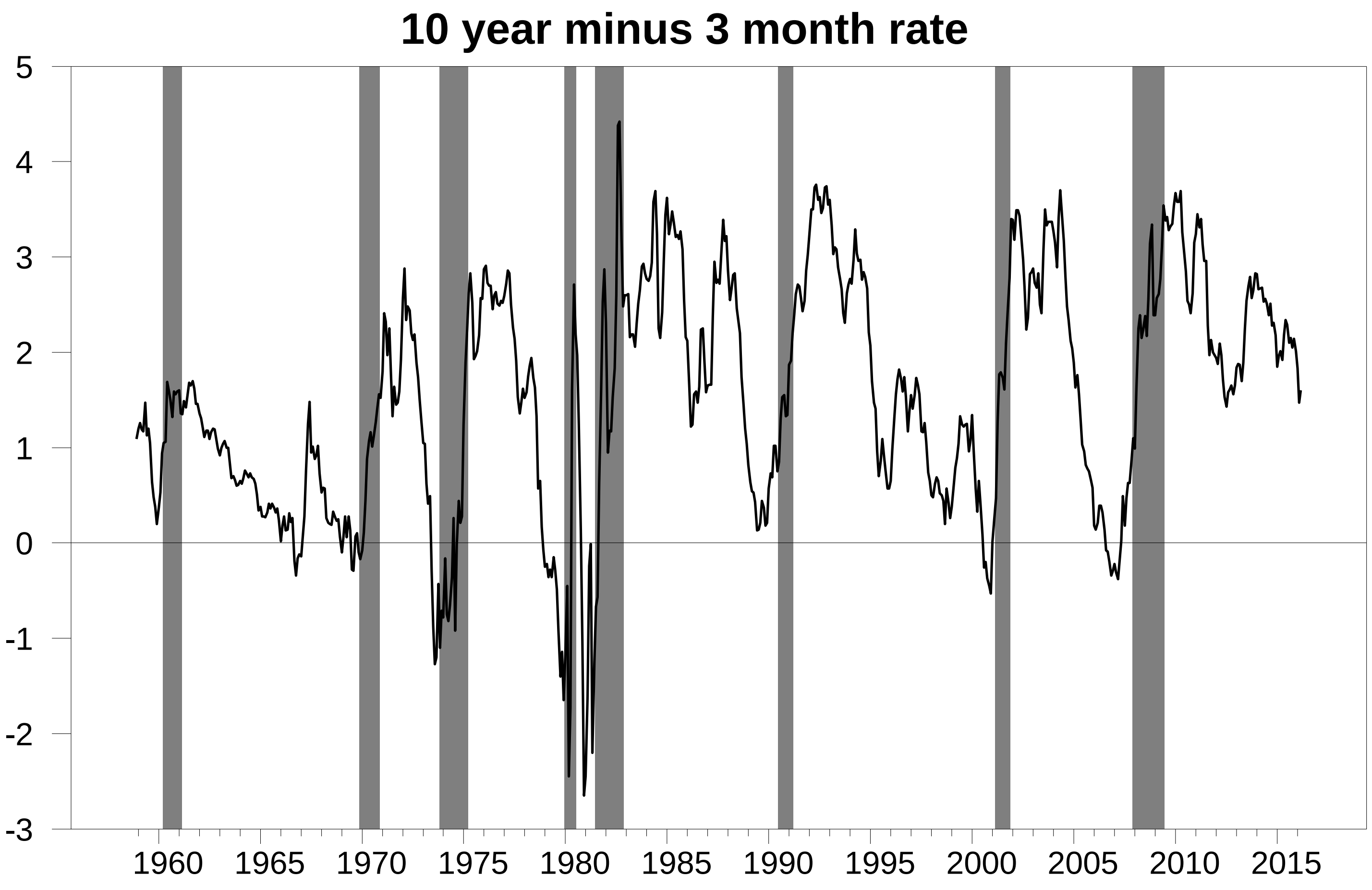
$$p_{nt} = -\int_0^n f_{xt} dx$$

which is known analytically.


Yield curve as of November 18, 2013
Yield to maturity
4.5
4
3.5
3
2.5
2
1.5
1
0.5
0
1
3
5
7
9
11
13
15
17
19
21
23
25
27
29
Maturity in years


Forward rates as of November 18, 2013
3.5
3
2.5
2
1.5
1
0.5
0
Jun 2014
Jun 2015
Jun 2016
Jun 2017
Jun 2018

Yield curve often inverts before recessions



Introduction to term structure of interest rates

A. Basic concepts
B. Expectations hypothesis of the term structure

Let $r_t = r_{1t} =$ risk-free one-period interest rate.

Option 1: lend \$1 today, have e^{r_t} next year.

Option 2: buy \$1 worth of n-year bonds $(= 1/P_{nt}$ units), sell next year (for $P_{n-1,t+1}$ per unit) $= P_{n-1,t+1}/P_{nt}$ dollars next year.

Risk neutral: expected return same:

$$e^{r_t} = E_t(P_{n-1,t+1}/P_{nt})$$

$$P_{nt} = e^{-r_t}E_t(P_{n-1,t+1})$$

$$P_{nt} = e^{-r_t} E_t(P_{n-1,t+1})$$

Claim: this implies

$$P_{nt} = E_t\{\exp[-(r_t + r_{t+1} + \cdots + r_{t+n-1})]\}$$

Proof: induction

(1) Holds for $n = 1$ by definition of r_t:

$$P_{1t} = E_t\{\exp[-r_t]\} \quad = e^{-r_t}.$$

(2) If holds for $n - 1$, then

$$\begin{aligned} P_{nt} &= e^{-r_t} E_t(P_{n-1,t+1}) \\ &= e^{-r_t} E_t[E_{t+1}\{\exp[-(r_{t+1} + \cdots + r_{t+n-1})]\}] \\ &= E_t[\{\exp[-(r_t + r_{t+1} + \cdots + r_{t+n-1})]\} \end{aligned}$$

$$P_{nt} = E_t[\{\exp[-(r_t + r_{t+1} + \cdots + r_{t+n-1})]\}$$

Jensen's Inequality:

$$P_{nt} \geq \exp[-E_t(r_t + r_{t+1} + \cdots + r_{t+n-1})]$$

e.g., if $\{r_{t+1}, \ldots, r_{t+n-1}\}$ are Gaussian,

$$P_{nt} = \exp\left[E_t\left(-\sum_{j=0}^{n-1} r_{t+j}\right) + (1/2)\mathrm{Var}_t\left(\sum_{j=0}^{n-1} r_{t+j}\right)\right]$$

$$p_{nt} = E_t\left(-\sum_{j=0}^{n-1} r_{t+j}\right) + (1/2)\mathrm{Var}_t\left(\sum_{j=0}^{n-1} r_{t+j}\right)$$

$$p_{nt} = E_t\left(-\sum_{j=0}^{n-1} r_{t+j}\right) + (1/2)\mathrm{Var}_t\left(\sum_{j=0}^{n-1} r_{t+j}\right)$$

or since $p_{nt} = -nr_{nt}$,

$$r_{nt} = n^{-1}E_t\left(\sum_{j=0}^{n-1} r_{t+j}\right) - (2n)^{-1}\mathrm{Var}_t\left(\sum_{j=0}^{n-1} r_{t+j}\right)$$

$$r_{nt} = n^{-1}E_t\left(\sum_{j=0}^{n-1} r_{t+j}\right) - (2n)^{-1}\text{Var}_t\left(\sum_{j=0}^{n-1} r_{t+j}\right)$$

Some researchers ignore the Jensen's Inequality term to interpret the Expectations Hypothesis of the term structure as

$$r_{nt} = n^{-1}E_t\left(\sum_{j=0}^{n-1} r_{t+j}\right)$$

or long rate is average expected future short rate (also called Log Pure Expectations Hypothesis).

Note that under LPEH,

$$p_{nt} = E_t\left(-\sum_{j=0}^{n-1} r_{t+j}\right)$$

$$p_{n+1,t} = E_t\left(-\sum_{j=0}^{n} r_{t+j}\right)$$

$$f_{nt} = p_{nt} - p_{n+1,t} = E_t(r_{t+n})$$

or forward rate is expected future short rate.

Introduction to term structure of interest rates

A. Basic concepts
B. Expectations hypothesis of the term structure
C. Response of the yield curve to news

(Gürkaynak, Sack, and Swanson, AER, 2005)

- Money Market Services surveyed private forecasters for value expected for important economic news releases
- Question: how does forward rate for different maturities *n* change in relation to the news release?

$$f_{nt} - f_{n,t-1} = \beta_{0n} + \sum_{j=1}^{J} \beta_{jn}(x_{jt} - \hat{x}_{jt}) + \varepsilon_{nt}$$

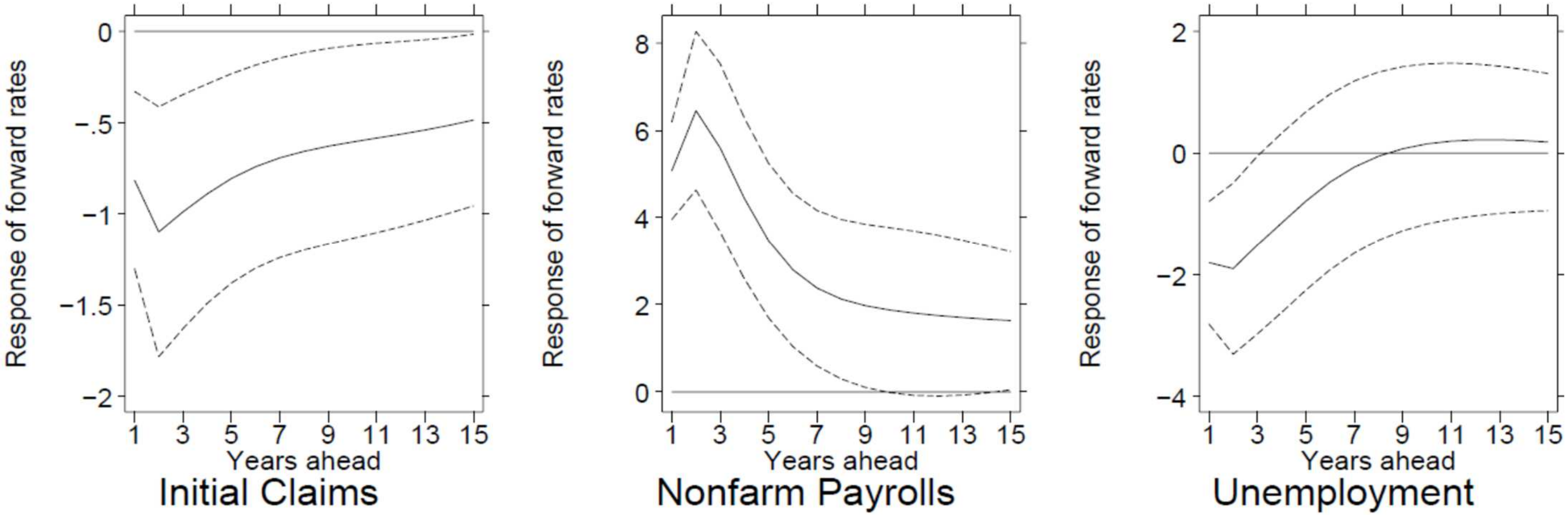
$f_{nt} - f_{n,t-1}$ = change in n-year instantaneous forward rate on day t

$\hat{x}_{jt}$ = market expectation of variable j

x_{jt} = actual released value

Plot β_{jn} as a function of n

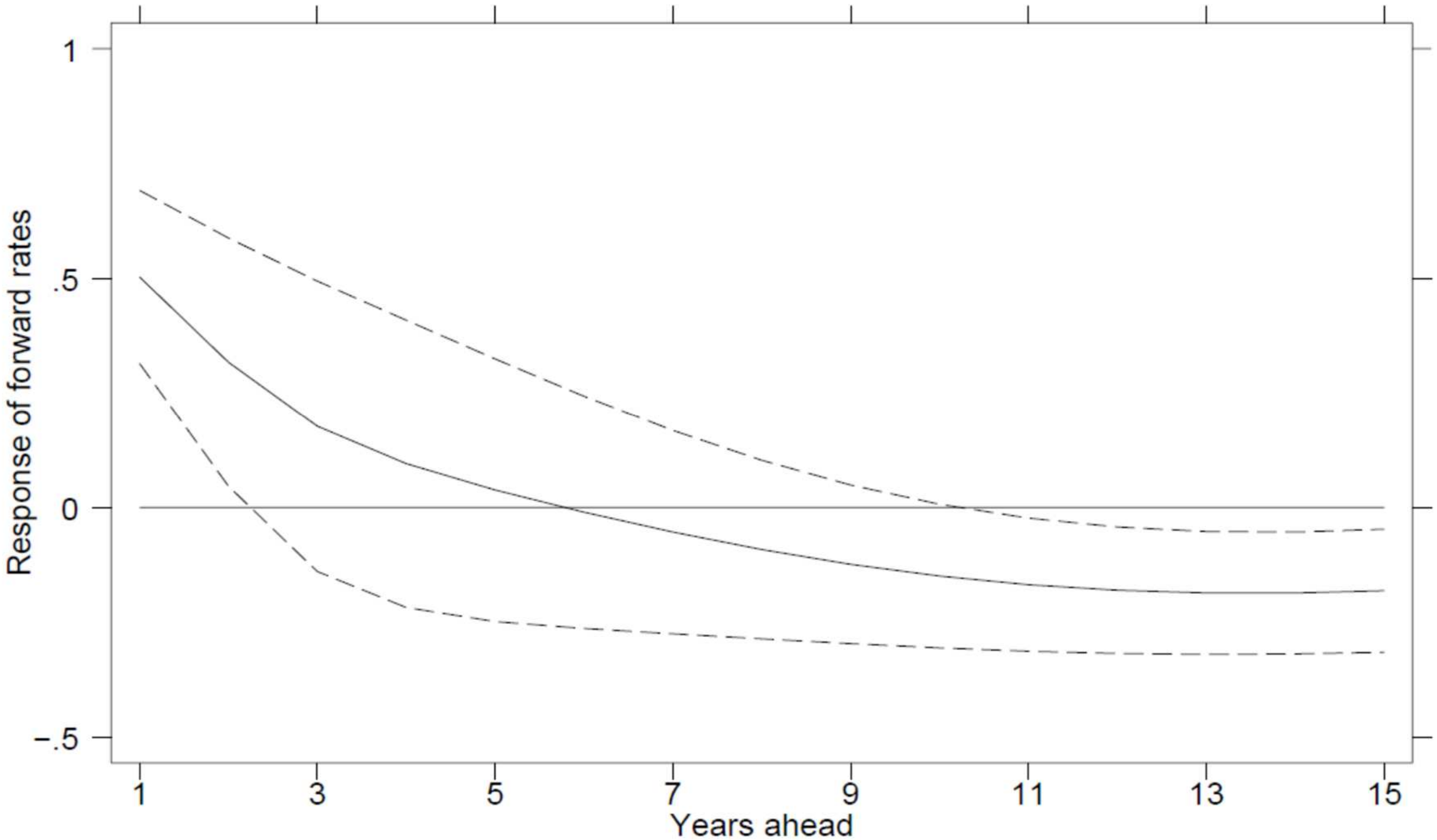
Response of forward rates with 95% confidence intervals (significant response after 7 years)



- Difficult to explain such persistent effects in calibrated DSGE
- Implausible long-run variation in real interest rate or persistence of fundamental shocks

- Additional evidence: how does yield curve change on day of monetary policy
- Regress change in forward rate on the unanticipated change in Fed’s target interest rate as inferred from fed funds futures

Response of forward rates to monetary policy surprises



- Surprising feature: long-term forward rate moves in *opposite* direction of short-term policy
- Fed raises short rate today but implies lower expected future short rates if we assume expectations hypothesis

GSS's explanation: Fed's long-run inflation target π_t^* is continually evolving.

- Explains response to macro news:
  - Higher employment temporarily raises inflation
  - Market expects Fed will make this permanent
- Explains response to monetary policy shocks:
  - Fed tightening today signals lower long-run inflation

- Treasury Inflation Protected Securities (TIPS)
  - coupon and yield rise with CPI
  - Market measure of real return
- Findings:
  - Real long forward rates do not respond to macro news, but expected inflation component does
  - Real long forward rates do not fall after monetary contraction, but expected inflation component does

Introduction to term structure of interest rates

A. Basic concepts
B. Expectations hypothesis of the term structure
C. Response of the yield curve to news
D. Risk aversion and the term structure

Although expectations hypothesis is convenient, it does not fit the data.

(1) Term structure usually slopes up: investor better off with long maturity.

(2) Excess holding yields are predictable.

LPEH:

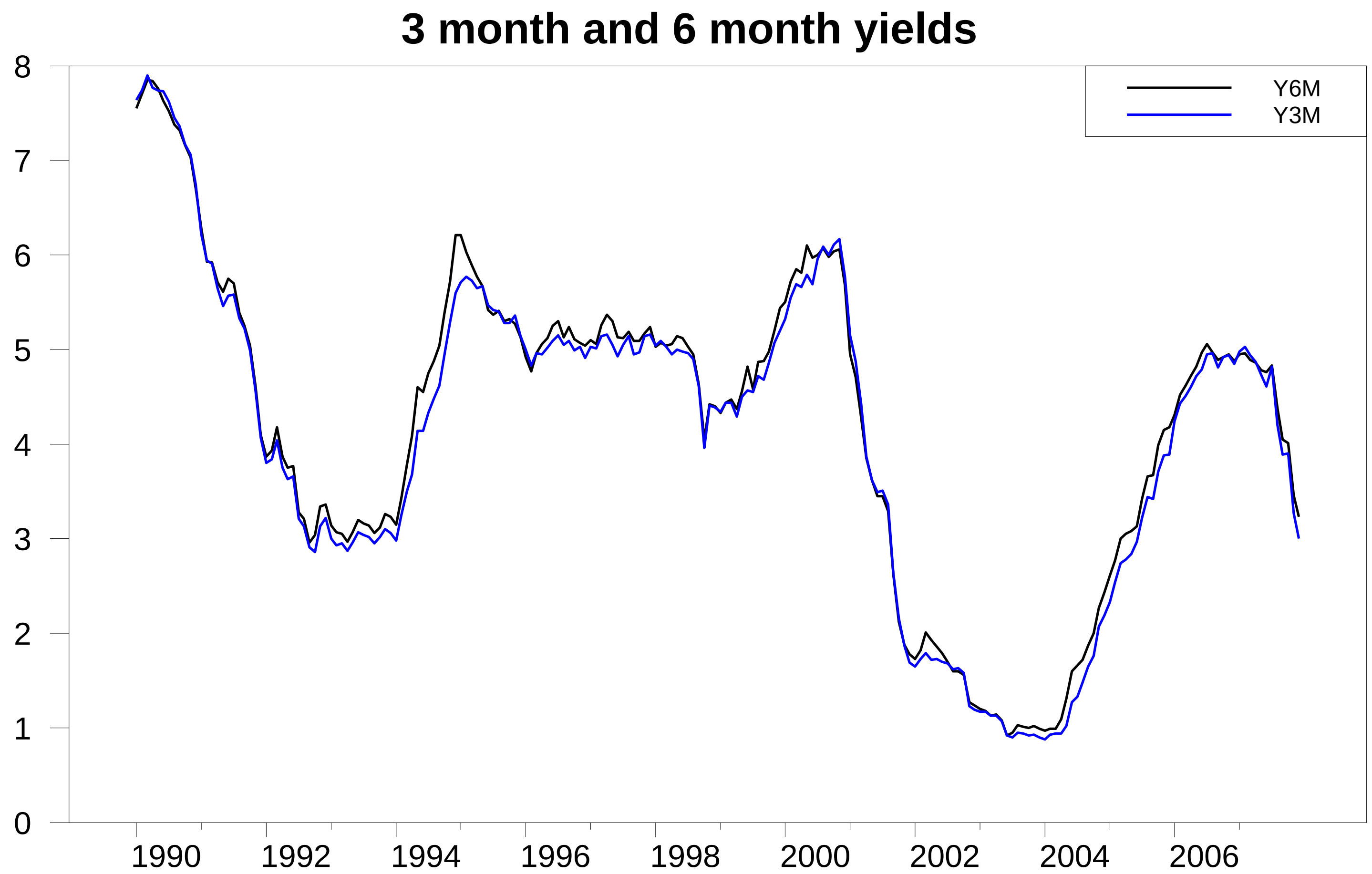
$$r_{2t} = (1/2)[r_{1t} + E_t(r_{1,t+1})]$$

$$2r_{2t} - r_{1t} = E_t(r_{1,t+1})$$

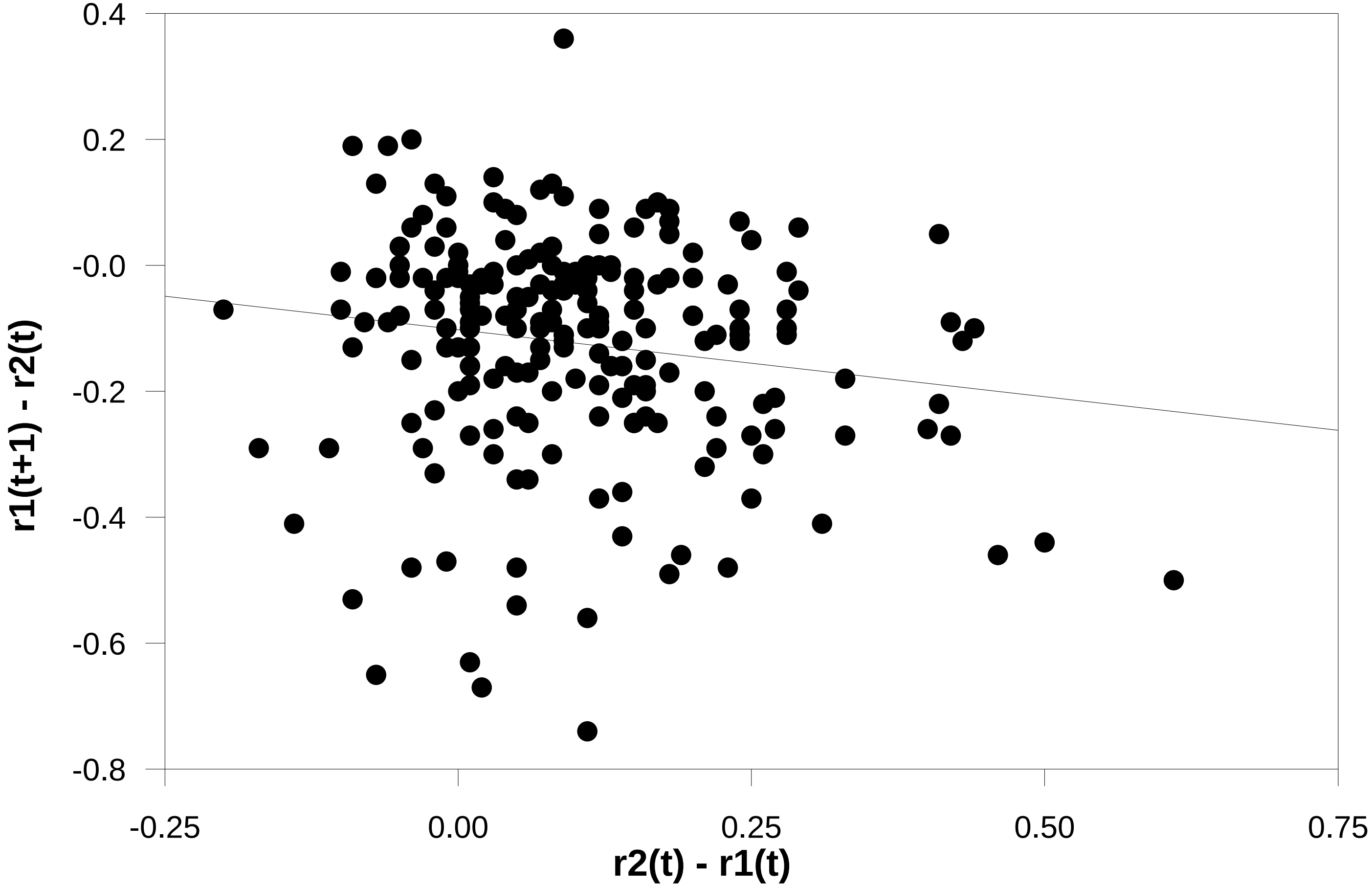
$$E_t(r_{1,t+1}) - r_{2t} = r_{2t} - r_{1t}$$

If 6-month yield is currently above 3-month $(r_{2t} - r_{1t} > 0)$, then next quarter's 3-month yield is expected to be higher than current 6 month $(r_{1,t+1} - r_{2t} > 0)$.

Opposite is usually observed



Regression of $r_{1,t+1} - r_{2t}$ on $r_{2t} - r_{1t}$ has wrong sign



We obtained expectations hypothesis by assuming risk-neutral investor. Consider instead someone with objective

$$E_t\left\{\sum_{j=0}^{\infty} \beta^j U(C_{t+j})\right\}$$

Invest \$1 in some asset k whose current nominal price is Q_{kt}, sell next period for $Q_{k,t+1}$.

Give up today $(1/P_t)$ units of C_t where P_t is the dollar price of consumption good:

$$\text{cost} = U'(C_t)/P_t$$

Next period gain $Q_{k,t+1}/(Q_{kt}P_{t+1})$ units of C_{t+1}

$$\text{gain} = E_t\left\{\frac{\beta U'(C_{t+1})Q_{k,t+1}}{Q_{kt}P_{t+1}}\right\}$$

Optimal choice by investor implies

$$\frac{U'(C_t)}{P_t} = E_t\left\{\frac{\beta U'(C_{t+1})Q_{k,t+1}}{Q_{kt}P_{t+1}}\right\}$$

$$\frac{U'(C_t)}{P_t} = E_t\left\{\frac{\beta U'(C_{t+1})Q_{k,t+1}}{Q_{kt}P_{t+1}}\right\}$$

$$1 = E_t\left\{\frac{M_{t+1}Q_{k,t+1}}{Q_{kt}}\right\}$$

$$M_{t+1} = \frac{\beta U'(C_{t+1})P_t}{U'(C_t)P_{t+1}}$$

for every asset k.

Applied to term structure of interest rates, price of n-period pure discount bond at t is P_{nt} and price of that asset at $t+1$ is $P_{n-1,t+1}$:

$$1 = E_t \left\{ \frac{M_{t+1} P_{n-1,t+1}}{P_{nt}} \right\}$$

for every maturity n.

M_{t+1} is called "pricing kernel" or "stochastic discount factor"

$$1 = E_t\left\{\frac{M_{t+1}P_{n-1,t+1}}{P_{nt}}\right\}$$

For $n = 1$,

$$1 = E_t\left\{\frac{M_{t+1}}{P_{1t}}\right\} = e^{r_t}E_t(M_{t+1})$$

$$E_t(M_{t+1}) = e^{-r_t}.$$

Risk-neutral investors (expectations hypothesis) is special case where

$$M_{t+1} = e^{-r_t}.$$

We derived the expression

$$1 = E_t\left\{\frac{M_{t+1}P_{n-1,t+1}}{P_{nt}}\right\}$$

by assuming a particular utility function (namely $E_t\left\{\sum_{j=0}^{\infty}\beta^j U(C_{t+j})\right\}$) which implied a particular value for M_{t+1} (namely $M_{t+1} = \frac{\beta U'(C_{t+1})P_t}{U'(C_t)P_{t+1}}$).

Alternatively, one can also deduce that

$$1 = E_t\left\{\frac{M_{t+1}P_{n-1,t+1}}{P_{nt}}\right\}$$

from a simple absence-of-arbitrage argument. If the equation does not hold for some M_{t+1} a function of the date $t+1$ state of the world, then there would be a way to buy some securities and sell others so as to generate positive cash flow at no cost.

Macro tradition: tries to look at particular model of investors to derive form for M_{t+1}.

Finance tradition: takes as given that there is some M_{t+1} and tries to describe its properties.