Forecasts and VARs

A. Intro to VARs
B. Nonorthogonalized IRF
C. Standard errors
D. Jordà local projections
E. Unit roots
F. Instability

A. Intro to VARs

Suppose we want to forecast y_{1t} based on:

$$y_{1,t-1}, y_{1,t-2}, \ldots, y_{1,t-p}$$

$$y_{2,t-1}, y_{2,t-2}, \ldots, y_{2,t-p}$$

$$\vdots$$

$$y_{n,t-1}, y_{n,t-2}, \ldots, y_{n,t-p}$$

deterministic functions of t

$(1, t, \cos(\pi t/6),$ seasonal dummies$)$

Let $\underset{(n\times 1)}{\mathbf{y}_t} = (y_{1t}, y_{2t}, \ldots, y_{nt})'$

$$\underset{(k\times 1)}{\mathbf{x}_t} = (1, \mathbf{y}'_{t-1}, \mathbf{y}'_{t-2}, \ldots, \mathbf{y}'_{t-p})'$$

$$k = np + 1$$

Suppose we consider linear forecast

$$\hat{y}_{1t|t-1} = \boldsymbol{\gamma}_1'\mathbf{x}_t$$

Best forecast within linear class:

value of $\boldsymbol{\gamma}_1$ that minimizes

$$E(y_{1t} - \boldsymbol{\gamma}_1'\mathbf{x}_t)^2$$

Proposition: If $\mathbf{y}_t$ is covariance-stationary and $E(\mathbf{x}_t\mathbf{x}_t')$ is nonsingular, then optimal forecast uses

$$\boldsymbol{\gamma}_1^* = E(\mathbf{x}_t\mathbf{x}_t')^{-1}E(\mathbf{x}_t y_t)$$

Definition: The optimal linear forecast,

$$\hat{y}_{1t|t-1} = \boldsymbol{\gamma}_1^{*'} \mathbf{x}_t,$$

is called the “population linear projection” of y_{1t} on $\mathbf{x}_t$

Definition: Ordinary least squares (OLS) estimate is given by

$$\hat{\boldsymbol{\gamma}}_1 = \left(\sum_{t=1}^{T} \mathbf{x}_t \mathbf{x}_t'\right)^{-1} \left(\sum_{t=1}^{T} \mathbf{x}_t y_{1t}\right)$$

Proposition: If $\mathbf{y}_t$ is stationary and ergodic, then

$$\hat{\boldsymbol{\gamma}}_1 \xrightarrow{p} \boldsymbol{\gamma}_1^*$$

Proof: (Law of Large Numbers)

$$\hat{\boldsymbol{\gamma}}_1 = \left(T^{-1} \sum_{t=1}^{T} \mathbf{x}_t \mathbf{x}_t' \right)^{-1} \left(T^{-1} \sum_{t=1}^{T} \mathbf{x}_t y_{1t} \right)$$

$$\xrightarrow{p} E(\mathbf{x}_t \mathbf{x}_t')^{-1} E(\mathbf{x}_t y_{1t})$$

If form separate forecasting equation

for each element of $\mathbf{y}_t$ and collect in vector,

$$\begin{aligned} y_{1t} &= \boldsymbol{\gamma}_1' \mathbf{x}_t + \varepsilon_{1t} \\ &\vdots \\ y_{nt} &= \boldsymbol{\gamma}_n' \mathbf{x}_t + \varepsilon_{nt} \\ \mathbf{y}_t &= \boldsymbol{\Gamma}' \mathbf{x}_t + \boldsymbol{\varepsilon}_t \end{aligned}$$

result is called vector autoregression:

$$\mathbf{y}_t = \mathbf{c} + \boldsymbol{\Phi}_1 \mathbf{y}_{t-1} + \boldsymbol{\Phi}_2 \mathbf{y}_{t-2} + \cdots + \boldsymbol{\Phi}_p \mathbf{y}_{t-p} + \boldsymbol{\varepsilon}_t$$

Above results imply we can consistently estimate coefficients for VAR by OLS equation by equation

$$\underset{(1\times k)}{\hat{\boldsymbol{\gamma}}_1'} = \left(\sum_{t=1}^{T} y_{1t}\mathbf{x}_t'\right)\left(\sum_{t=1}^{T} \mathbf{x}_t\mathbf{x}_t'\right)^{-1}$$

$$\vdots$$

$$\underset{(1\times k)}{\hat{\boldsymbol{\gamma}}_n'} = \left(\sum_{t=1}^{T} y_{nt}\mathbf{x}_t'\right)\left(\sum_{t=1}^{T} \mathbf{x}_t\mathbf{x}_t'\right)^{-1}$$

$$\underset{(n\times k)}{\hat{\boldsymbol{\Gamma}}'} = \left(\sum_{t=1}^{T} \mathbf{y}_t\mathbf{x}_t'\right)\left(\sum_{t=1}^{T} \mathbf{x}_t\mathbf{x}_t'\right)^{-1}$$

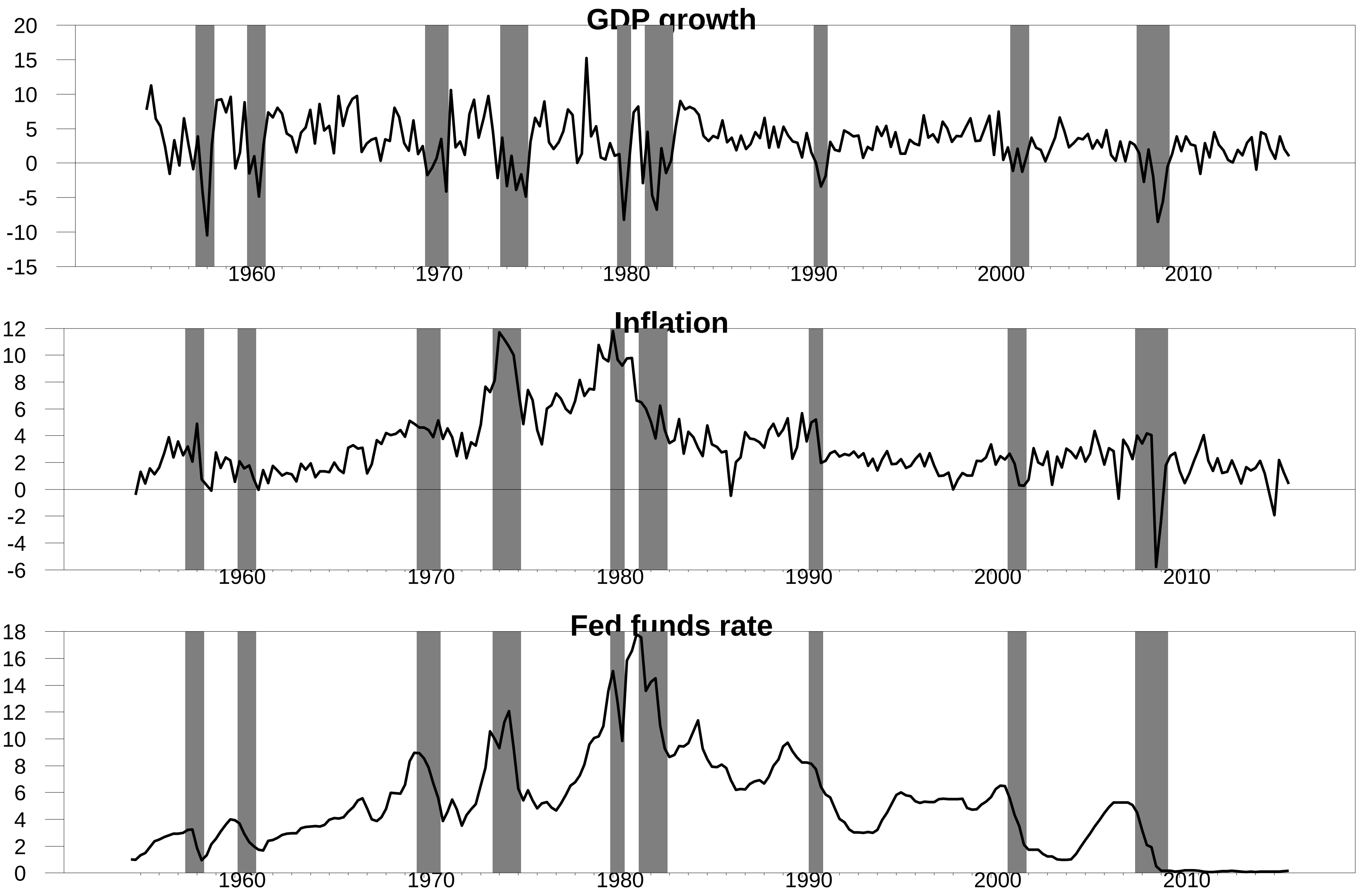
$$\hat{\boldsymbol{\Gamma}}' = \begin{bmatrix} \widehat{\mathbf{c}} & \hat{\boldsymbol{\Phi}}_1 & \hat{\boldsymbol{\Phi}}_2 & \cdots & \hat{\boldsymbol{\Phi}}_p \end{bmatrix}$$

Example

$$y_{1t} = 400 \times \text{quarterly log change in real GDP}$$

$$y_{2t} = 400 \times \text{quarterly log change in PCE deflator}$$

$$y_{3t} = \text{average fed funds rate over quarter}$$


GDP growth
20
15
10
5
0
-5
-10
-15
1960
1970
1980
1990
2000
2010
Inflation
12
10
8
6
4
2
0
-2
-4
-6
1960
1970
1980
1990
2000
2010
Fed funds rate
18
16
14
12
10
8
6
4
2
0
1960
1970
1980
1990
2000
2010

- Estimate with 4 lags on each variable for 1960:Q1 to 1990:Q4.
- Data and code to replicate provided at course webpage.
- Sample code shows how to compare 4 versus 5 lags using hypothesis tests, AIC, or BIC.
- See Lütkepohl, Section 4.3 for description.

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VAR/System - Estimation by Least Squares
Quarterly Data From 1960:01 To 1990:04
Usable Observations                          124

Dependent Variable GDPCH
Mean of Dependent Variable        3.4518389246
Std Error of Dependent Variable   3.8882938217
Standard Error of Estimate        3.3995169372
Sum of Squared Residuals          1282.7954101
Durbin-Watson Statistic                 1.9156
```

Variable	Coeff	Std Error	T-Stat
1. GDPCH{1}	0.140889900	0.096939607	1.45338
2. GDPCH{2}	0.171458188	0.095324153	1.79869
3. GDPCH{3}	0.019889121	0.095968000	0.20725
4. GDPCH{4}	0.027744892	0.088859228	0.31223
5. INFLATION{1}	-0.136199595	0.255977819	-0.53208
6. INFLATION{2}	0.109623520	0.293744691	0.37319
7. INFLATION{3}	0.019771719	0.294628089	0.06711
8. INFLATION{4}	-0.002777166	0.265583119	-0.01046
9. FEDFUNDS{1}	0.073563677	0.324268492	0.22686
10. FEDFUNDS{2}	-1.515272886	0.434675848	-3.48598
11. FEDFUNDS{3}	1.135868980	0.459482934	2.47206
12. FEDFUNDS{4}	-0.002977962	0.340933515	-0.00873
13. Constant	4.458801828	1.322775244	3.37079

```
Dependent Variable INFLATION
Mean of Dependent Variable       4.4308450671
Std Error of Dependent Variable  2.7295747540
Standard Error of Estimate       1.1933641353
Sum of Squared Residuals         158.07709350
Durbin-Watson Statistic                1.9869
```

Variable	Coeff	Std Error	T-Stat
1. GDPCH{1}	0.010633189	0.034029614	0.31247
2. GDPCH{2}	-0.017609241	0.033462527	-0.52624
3. GDPCH{3}	-0.024943566	0.033688542	-0.74042
4. GDPCH{4}	0.134451472	0.031193084	4.31030
5. INFLATION{1}	0.639779521	0.089858281	7.11987
6. INFLATION{2}	0.164641352	0.103115938	1.59666
7. INFLATION{3}	0.210029088	0.103426046	2.03072
8. INFLATION{4}	-0.043699446	0.093230119	-0.46873
9. FEDFUNDS{1}	0.162052487	0.113830993	1.42362
10. FEDFUNDS{2}	-0.195321038	0.152588317	-1.28005
11. FEDFUNDS{3}	-0.065974240	0.161296580	-0.40902
12. FEDFUNDS{4}	0.076340807	0.119681071	0.63787
13. Constant	-0.046316742	0.464346131	-0.09975

```
Dependent Variable FEDFUNDS
Mean of Dependent Variable       7.1162903226
Std Error of Dependent Variable  3.4537135588
Standard Error of Estimate       1.0084648414
Sum of Squared Residuals         112.88714833
Durbin-Watson Statistic                2.0215

    Variable                            Coeff       Std Error      T-Stat
*************************************************************************
1.  GDPCH{1}                         0.092721729  0.028757081     3.22431
2.  GDPCH{2}                         0.055587277  0.028277858     1.96575
3.  GDPCH{3}                         0.016988953  0.028468855     0.59676
4.  GDPCH{4}                         0.006331597  0.026360042     0.24020
5.  INFLATION{1}                     0.227483091  0.075935680     2.99573
6.  INFLATION{2}                     0.066210929  0.087139202     0.75983
7.  INFLATION{3}                     0.007176875  0.087401262     0.08211
8.  INFLATION{4}                    -0.146731250  0.078785087    -1.86242
9.  FEDFUNDS{1}                      0.982722020  0.096194071    10.21604
10. FEDFUNDS{2}                     -0.451940303  0.128946353    -3.50487
11. FEDFUNDS{3}                      0.502225982  0.136305361     3.68457
12. FEDFUNDS{4}                     -0.102751117  0.101137741    -1.01595
13. Constant                        -0.752466690  0.392400553    -1.91760
```

B. Nonorthogonalized IRF

$$\mathbf{y}_{t+1} = \mathbf{c} + \mathbf{\Phi}_1\mathbf{y}_t + \mathbf{\Phi}_2\mathbf{y}_{t-1} + \cdots + \mathbf{\Phi}_p\mathbf{y}_{t-p+1} + \boldsymbol{\varepsilon}_{t+1}$$

$$\mathbf{y}_{t+2} = \mathbf{c} + \mathbf{\Phi}_1\mathbf{y}_{t+1} + \mathbf{\Phi}_2\mathbf{y}_t + \cdots + \mathbf{\Phi}_p\mathbf{y}_{t-p+2} + \boldsymbol{\varepsilon}_{t+2}$$

$$\mathbf{y}_{t+2} = \mathbf{c} + \mathbf{\Phi}_1\left[\mathbf{c} + \mathbf{\Phi}_1\mathbf{y}_t + \mathbf{\Phi}_2\mathbf{y}_{t-1} + \cdots + \mathbf{\Phi}_p\mathbf{y}_{t-p+1} + \boldsymbol{\varepsilon}_{t+1}\right]$$

$$+ \mathbf{\Phi}_2\mathbf{y}_t + \cdots + \mathbf{\Phi}_p\mathbf{y}_{t-p+1} + \boldsymbol{\varepsilon}_{t+2}$$

$$= \mathbf{c}_2 + \mathbf{\Psi}_0\boldsymbol{\varepsilon}_{t+2} + \mathbf{\Psi}_1\boldsymbol{\varepsilon}_{t+1} + \mathbf{\Psi}_2\mathbf{y}_t + \mathbf{H}_{22}\mathbf{y}_{t-1} + \cdots + \mathbf{H}_{2p}\mathbf{y}_{t-p+1}$$

$$\mathbf{\Psi}_0 = \mathbf{I}_n$$

$$\mathbf{\Psi}_1 = \mathbf{\Phi}_1$$

$$\mathbf{\Psi}_2 = \mathbf{\Phi}_1^2 + \mathbf{\Phi}_2$$

$$\mathbf{y}_{t+2} = \mathbf{c}_2 + \mathbf{\Psi}_0\boldsymbol{\varepsilon}_{t+2} + \mathbf{\Psi}_1\boldsymbol{\varepsilon}_{t+1} + \mathbf{\Psi}_2\mathbf{y}_t + \mathbf{H}_{22}\mathbf{y}_{t-1} + \cdots + \mathbf{H}_{2p}\mathbf{y}_{t-p+1}$$

We know that $\boldsymbol{\varepsilon}_{t+1}$ is uncorrelated with $\mathbf{y}_t, \ldots, \mathbf{y}_{t-p+1}$ by definition of the plim.

If VAR has enough lags, $\boldsymbol{\varepsilon}_{t+2}$ is also uncorrelated with $\mathbf{y}_t, \ldots, \mathbf{y}_{t-p+1}$.

$$\Rightarrow \hat{\mathbf{y}}_{t+2|t} = \mathbf{c}_2 + \mathbf{\Psi}_2\mathbf{y}_t + \mathbf{H}_{22}\mathbf{y}_{t-1} + \cdots + \mathbf{H}_{2p}\mathbf{y}_{t-p+1}$$

$$\frac{\partial \hat{\mathbf{y}}_{t+2|t}}{\partial \mathbf{y}_t'} = \mathbf{\Psi}_2$$

$$\mathbf{y}_{t+s} = \mathbf{c}_s + \boldsymbol{\Psi}_0\boldsymbol{\varepsilon}_{t+s} + \boldsymbol{\Psi}_1\boldsymbol{\varepsilon}_{t+s-1} + \boldsymbol{\Psi}_2\boldsymbol{\varepsilon}_{t+s-2} + \cdots + \boldsymbol{\Psi}_{s-1}\boldsymbol{\varepsilon}_{t+1}$$

$$+ \boldsymbol{\Psi}_s\mathbf{y}_t + \mathbf{H}_{s2}\mathbf{y}_{t-1} + \cdots + \mathbf{H}_{sp}\mathbf{y}_{t-p+1}$$

$$\frac{\partial\hat{\mathbf{y}}_{t+s|t}}{\partial\mathbf{y}_t'} = \boldsymbol{\Psi}_s$$

$$\boldsymbol{\Psi}_0 = \mathbf{I}_n$$

$$\boldsymbol{\Psi}_1 = \boldsymbol{\Phi}_1$$

$$\boldsymbol{\Psi}_2 = \boldsymbol{\Phi}_1^2 + \boldsymbol{\Phi}_2$$

$$\boldsymbol{\Psi}_s = \boldsymbol{\Phi}_1\boldsymbol{\Psi}_{s-1} + \boldsymbol{\Phi}_2\boldsymbol{\Psi}_{s-2} + \cdots + \boldsymbol{\Phi}_p\boldsymbol{\Psi}_{s-p}$$

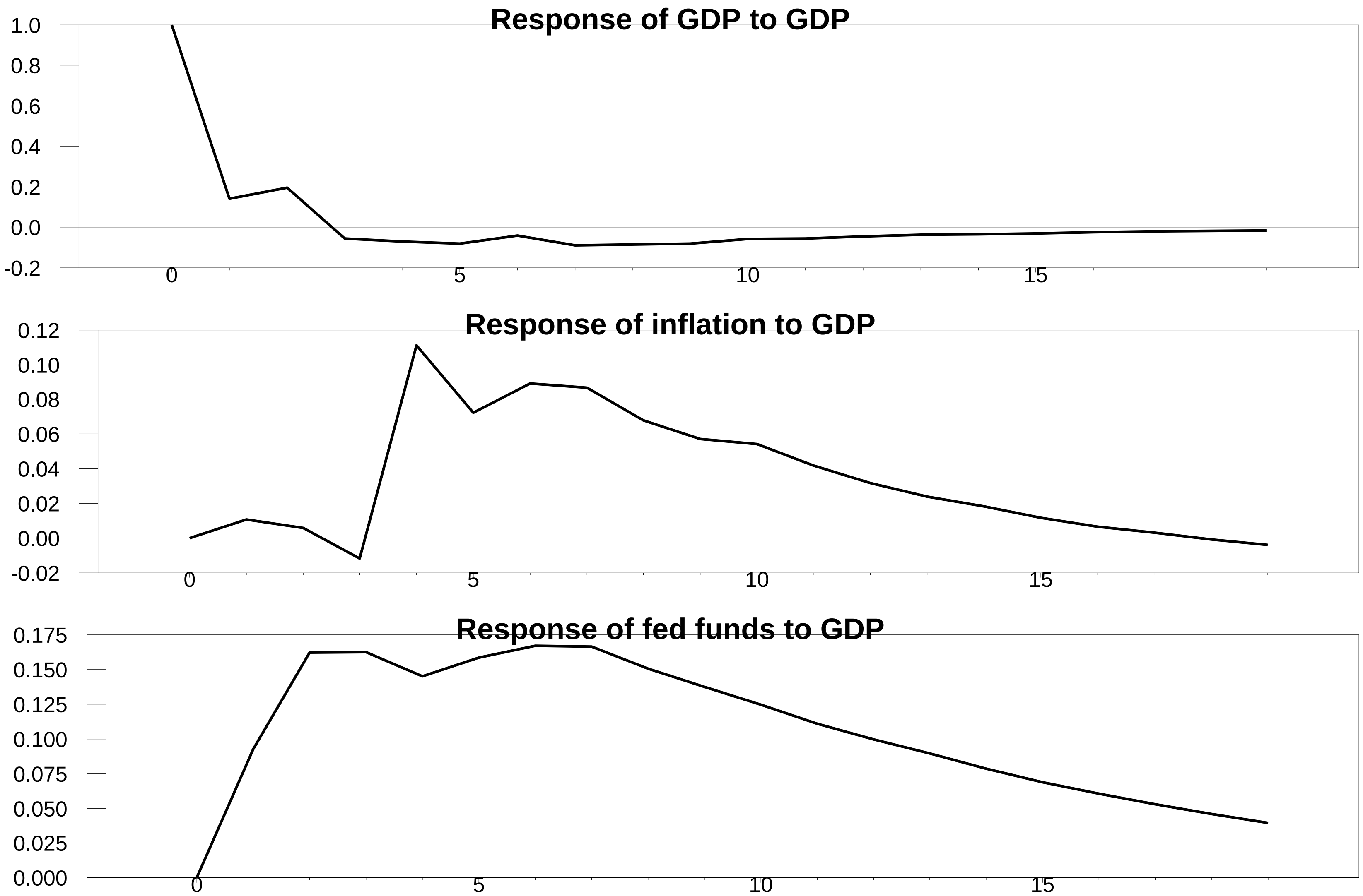
$$\boldsymbol{\Psi}_s = \boldsymbol{\Phi}_1 \boldsymbol{\Psi}_{s-1} + \boldsymbol{\Phi}_2 \boldsymbol{\Psi}_{s-2} + \cdots + \boldsymbol{\Phi}_p \boldsymbol{\Psi}_{s-p}$$

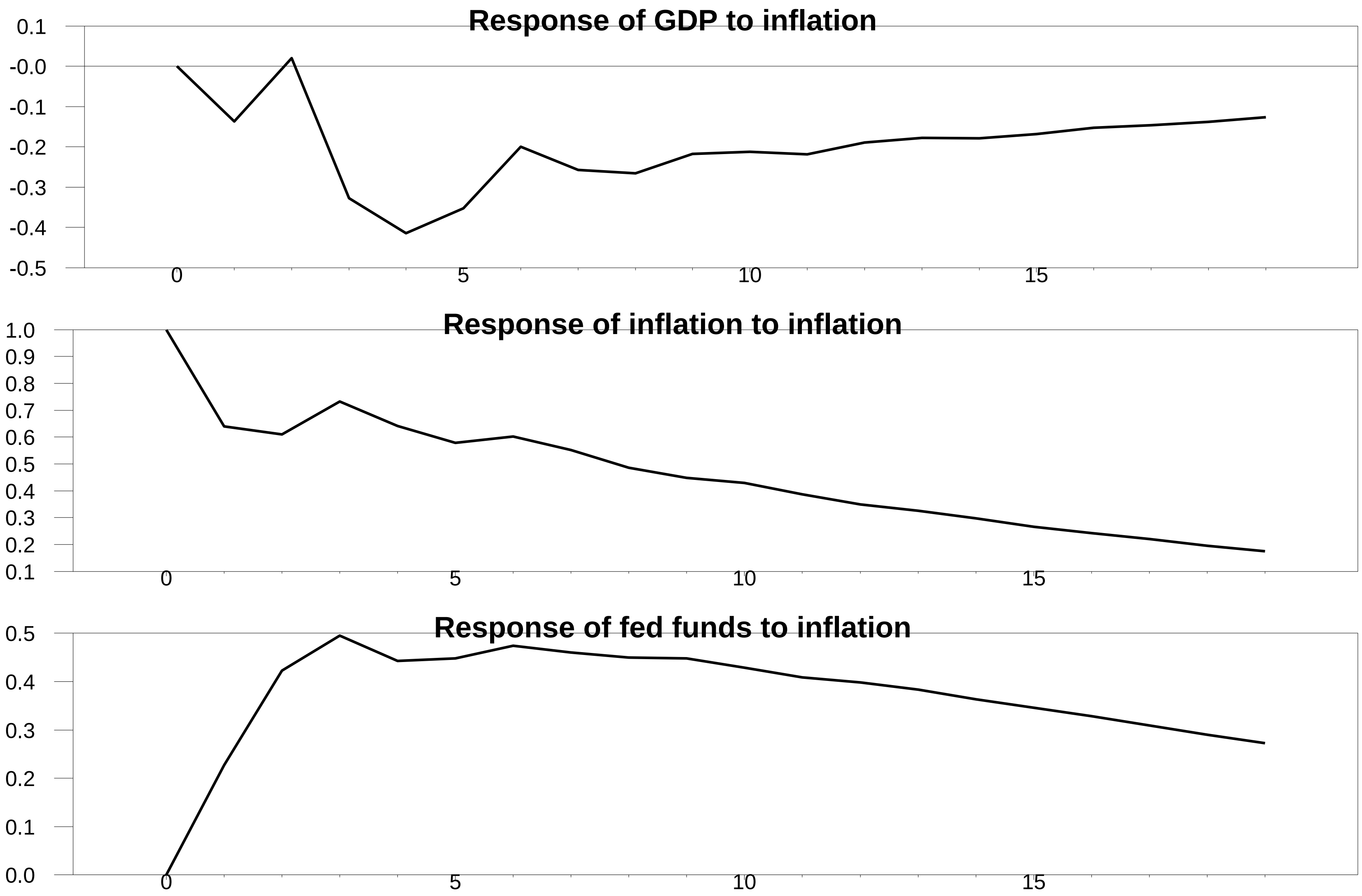
Column j of $\boldsymbol{\Psi}_s$ is answer to the question:
How does my forecast of $\mathbf{y}_{t+s}$ change if I increase y_{jt} by one unit holding all other elements of $\mathbf{y}_t$ and all elements of $\mathbf{y}_{t-1}, \ldots, \mathbf{y}_{t-p}$ constant.

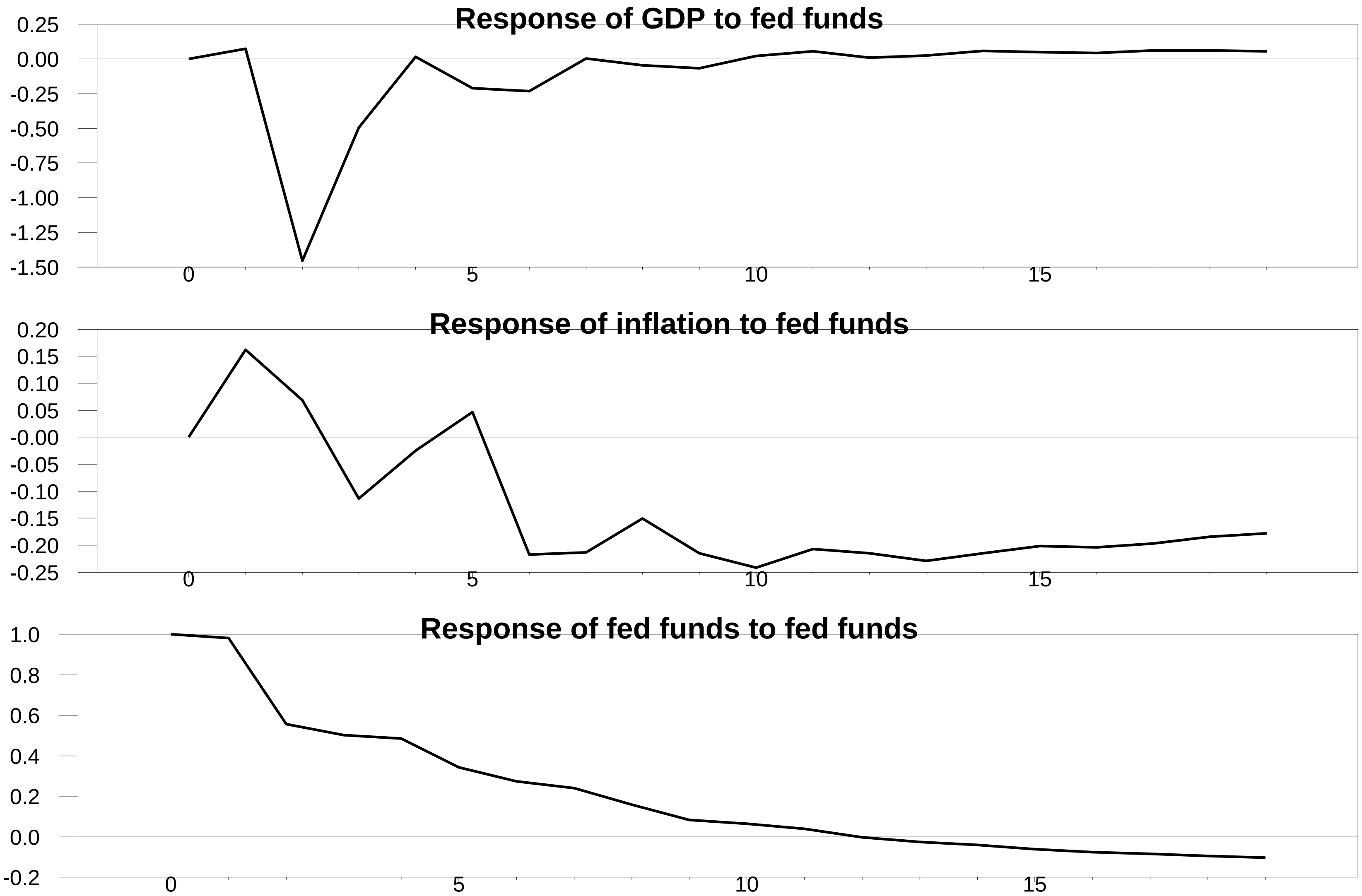
The sequence of $n \times n$ matrices $\{\boldsymbol{\Psi}_s\}_{s=0,1,2,\ldots}$ is called the nonorthogonalized impulse-response function.

Responses to Shock in GDPCH

Entry	GDPCH	INFLATION	FEDFUNDS
1	1.0000000	0.0000000	0.0000000
2	0.1408899	0.0106332	0.0927217
3	0.1966809	0.0057176	0.1621894
4	-0.0564244	-0.0117519	0.1625443
5	-0.0697265	0.1110743	0.1452111
6	-0.0796774	0.0723151	0.1587723
7	-0.0409759	0.0890917	0.1672757
8	-0.0881861	0.0867922	0.1666632
9	-0.0851984	0.0678733	0.1506179
10	-0.0802485	0.0571617	0.1376026
11	-0.0569765	0.0540731	0.1247829
12	-0.0546655	0.0416753	0.1110460
13	-0.0452988	0.0316166	0.0997375
14	-0.0375426	0.0238060	0.0895958
15	-0.0336034	0.0181658	0.0785062
16	-0.0299729	0.0115640	0.0688406
17	-0.0236107	0.0066515	0.0607137
18	-0.0204608	0.0030549	0.0530267
19	-0.0181050	-0.0007017	0.0457998
20	-0.0150533	-0.0040129	0.0394487


Response of GDP to GDP
1.0
0.8
0.6
0.4
0.2
0.0
-0.2
0
5
10
15
Response of inflation to GDP
0.12
0.10
0.08
0.06
0.04
0.02
0.00
-0.02
0
5
10
15
Response of fed funds to GDP
0.175
0.150
0.125
0.100
0.075
0.050
0.025
0.000
0
5
10
15


Response of GDP to inflation
0.1
-0.0
-0.1
-0.2
-0.3
-0.4
-0.5
0
5
10
15
Response of inflation to inflation
1.0
0.9
0.8
0.7
0.6
0.5
0.4
0.3
0.2
0.1
0
5
10
15
Response of fed funds to inflation
0.5
0.4
0.3
0.2
0.1
0.0
0
5
10
15



Response of GDP to fed funds
0.25
0.00
-0.25
-0.50
-0.75
-1.00
-1.25
-1.50
0
5
10
15
Response of inflation to fed funds
0.20
0.15
0.10
0.05
-0.00
-0.05
-0.10
-0.15
-0.20
-0.25
0
5
10
15
Response of fed funds to fed funds
1.0
0.8
0.6
0.4
0.2
0.0
-0.2
0
5
10
15

C. Standard errors

Generate standard errors using Bayesian posterior distribution based on diffuse priors.

$$\underset{(n\times 1)}{\mathbf{y}_t} = (y_{1t}, y_{2t}, \ldots, y_{nt})'$$

$$\underset{(k\times 1)}{\mathbf{x}_t} = (1, \mathbf{y}'_{t-1}, \mathbf{y}'_{t-2}, \ldots, \mathbf{y}'_{t-p})'$$

$$k = np + 1$$

$$\mathbf{y}_t = \boldsymbol{\Gamma}'\mathbf{x}_t + \boldsymbol{\varepsilon}_t$$

$$E(\boldsymbol{\varepsilon}_t\boldsymbol{\varepsilon}'_t) = \boldsymbol{\Omega}$$

$$\underset{(n\times k)}{\hat{\boldsymbol{\Gamma}}'} = \left(\sum_{t=1}^{T} \mathbf{y}_t \mathbf{x}_t'\right)\left(\sum_{t=1}^{T} \mathbf{x}_t \mathbf{x}_t'\right)^{-1}$$

$$\underset{(n\times 1)}{\hat{\boldsymbol{\varepsilon}}_t} = \mathbf{y}_t - \hat{\boldsymbol{\Gamma}}' \mathbf{x}_t$$

$$\underset{(n\times n)}{\hat{\boldsymbol{\Omega}}} = T^{-1} \sum_{t=1}^{T} \hat{\boldsymbol{\varepsilon}}_t \hat{\boldsymbol{\varepsilon}}_t'$$

$\mathbf{\Omega}^{-1}|\mathbf{y}_1, \ldots, \mathbf{y}_T \sim$ Wishart with $T - p$ degrees of freedom and scale matrix $T\hat{\mathbf{\Omega}}$

$$\text{Wishart}(k, \mathbf{H}) = \mathbf{z}_1\mathbf{z}_1' + \cdots + \mathbf{z}_k\mathbf{z}_k'$$

$$\underset{(n\times 1)}{\mathbf{z}_i} \sim N(\mathbf{0}, \mathbf{H}^{-1})$$

$$\text{vec}(\boldsymbol{\Gamma})|\boldsymbol{\Omega}, \mathbf{y}_1, \ldots, \mathbf{y}_T \sim N\left(\text{vec}\left[\hat{\boldsymbol{\Gamma}}\right], \boldsymbol{\Omega} \otimes \left[\sum_{t=1}^{T} \mathbf{x}_t \mathbf{x}_t'\right]^{-1}\right)$$

(1) Draw $\boldsymbol{\Omega}^{(m)}$ and $\boldsymbol{\Gamma}^{(m)}$ for this distribution

(2) For each $m = 1, \ldots, 10^4$ calculate $\boldsymbol{\Psi}_s^{(m)}$

(3) For each i, j, s find 95% interval for row i col j element of this matrix

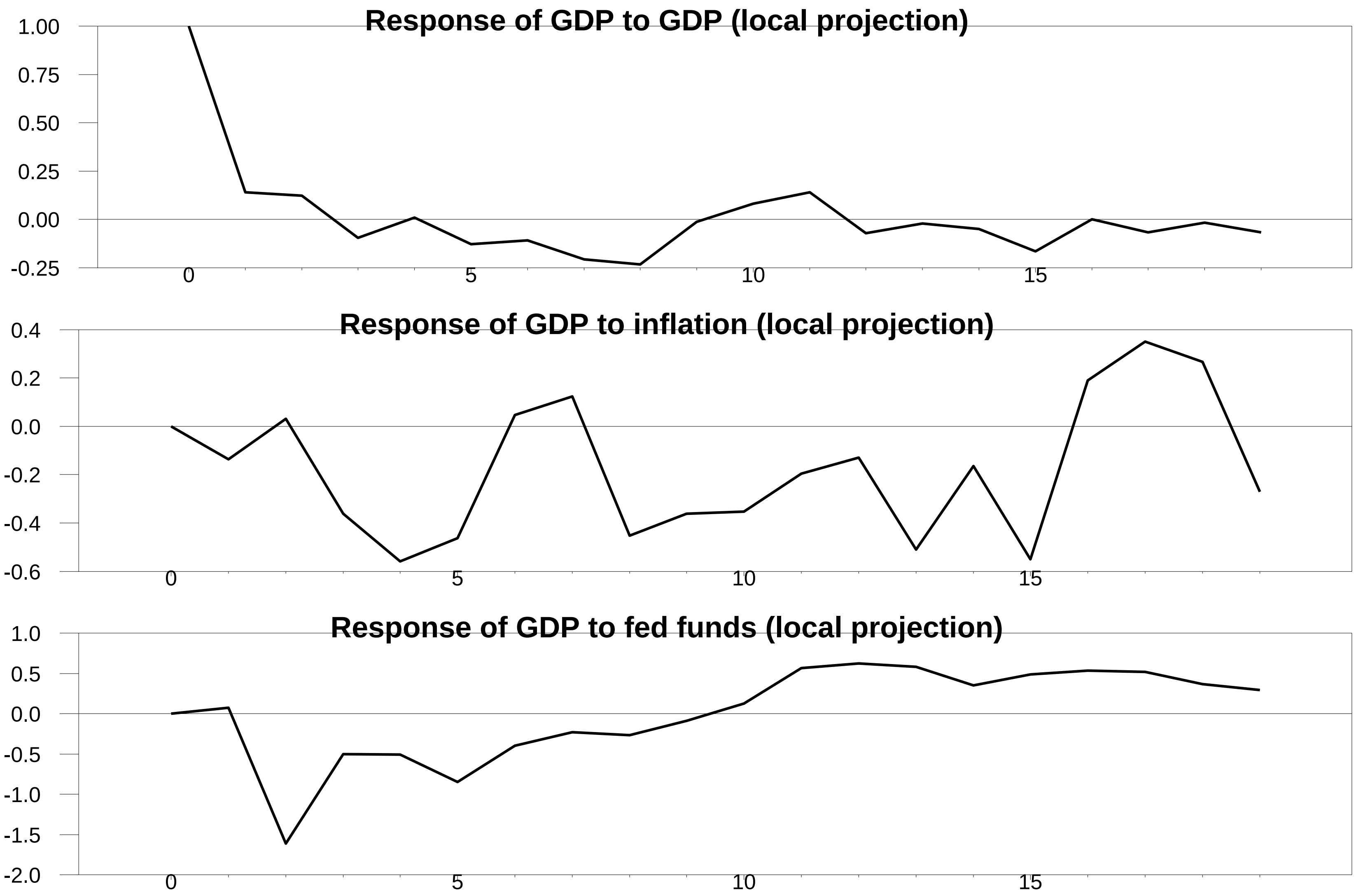
D. Jordà local projections

As noted by Jordà (2005), we can also estimate forecast without imposing VAR structure.

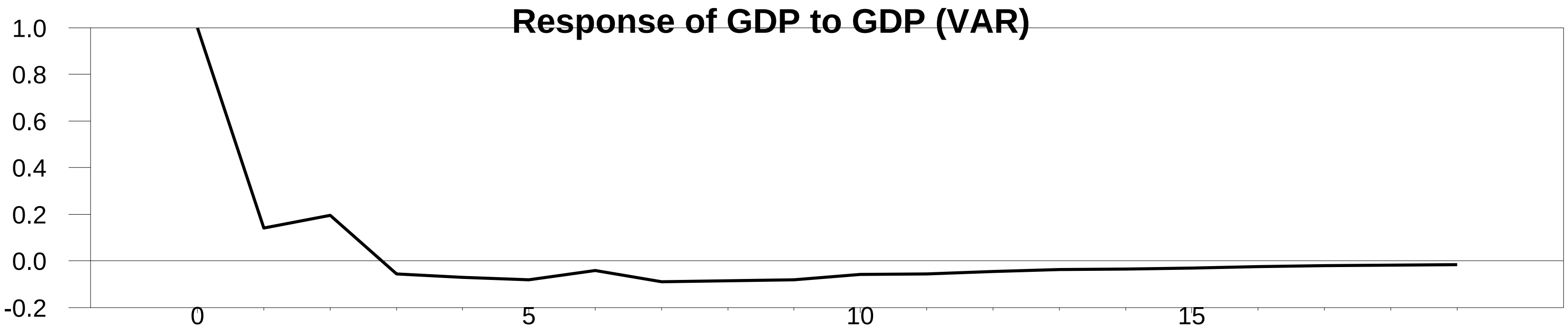
$$\mathbf{y}_{t+s} = \mathbf{c}_s + \mathbf{\Psi}_s \mathbf{y}_t + \mathbf{H}_{s2} \mathbf{y}_{t-1} + \cdots + \mathbf{H}_{sp} \mathbf{y}_{t-p+1} + \mathbf{u}_{t+s}$$

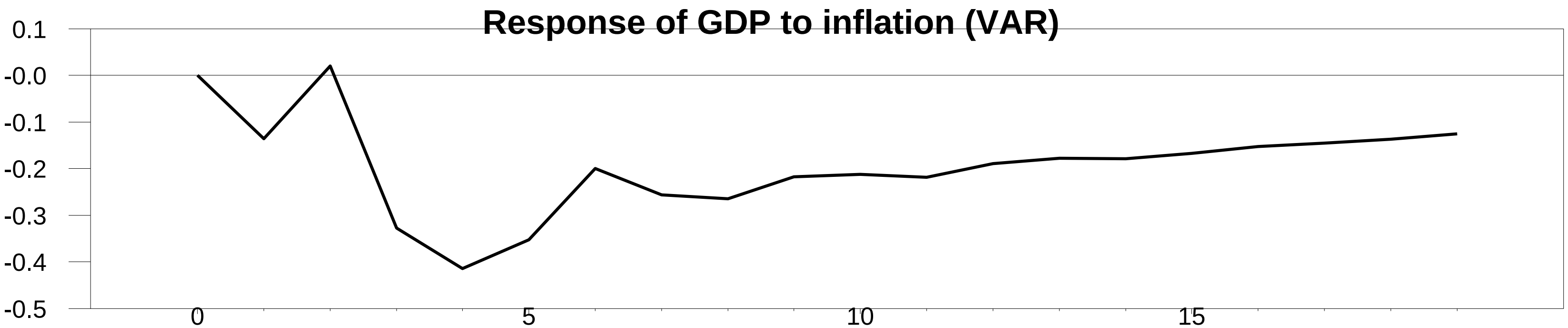
Estimate by n different regressions separately for each s.

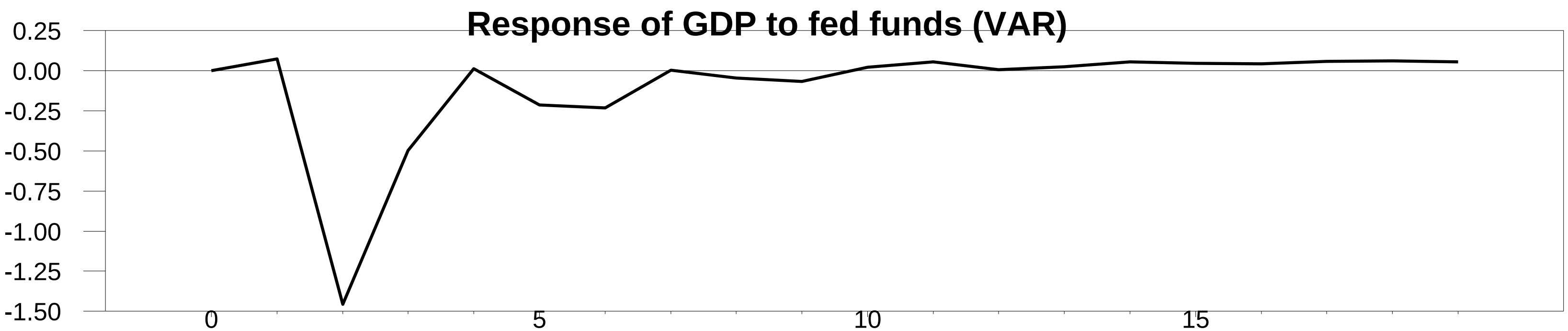
Resulting $\{\hat{\mathbf{\Psi}}_s\}_{s=1,2,...}$ is direct estimate of nonorthogonalized IRF.



Response of GDP to GDP (local projection)
1.00
0.75
0.50
0.25
0.00
-0.25
0
5
10
15
Response of GDP to inflation (local projection)
0.4
0.2
0.0
-0.2
-0.4
-0.6
0
5
10
15
Response of GDP to fed funds (local projection)
1.0
0.5
0.0
-0.5
-1.0
-1.5
-2.0
0
5
10
15


Response of GDP to GDP (VAR)
1.0
0.8
0.6
0.4
0.2
0.0
-0.2
0
5
10
15


Response of GDP to inflation (VAR)
0.1
-0.0
-0.1
-0.2
-0.3
-0.4
-0.5
0
5
10
15


Response of GDP to fed funds (VAR)
0.25
0.00
-0.25
-0.50
-0.75
-1.00
-1.25
-1.50
0
5
10
15

- Local projections and VAR recursion should give similar broad picture.
- Local projections likely more volatile and may give worse forecasts (Marcellino, Stock and Watson, 2006)

E. Unit roots

In exercises so far we took y_{1t} to be growth rate of real GDP.

What if we had instead used the level of GDP without differencing?

Coefficients for GDP equation when estimated in growth rates versus levels

Growth rate regression			Levels regression	
GDPCH{1}	0.14089		GDPLOG{1}	1.125783
GDPCH{2}	0.171458		GDPLOG{2}	0.021619
GDPCH{3}	0.019889		GDPLOG{3}	-0.14394
GDPCH{4}	0.027745		GDPLOG{4}	0.000463
INFLATION{1}	-0.1362		INFLATION{1}	-0.1873
INFLATION{2}	0.109624		INFLATION{2}	0.101033
INFLATION{3}	0.019772		INFLATION{3}	0.023217
INFLATION{4}	-0.00278		INFLATION{4}	0.033663
FEDFUNDS{1}	0.073564		FEDFUNDS{1}	0.091929
FEDFUNDS{2}	-1.51527		FEDFUNDS{2}	-1.53255
FEDFUNDS{3}	1.135869		FEDFUNDS{3}	1.127251
FEDFUNDS{4}	-0.00298		FEDFUNDS{4}	-0.11631
Constant	4.458802		Constant	-7.76291

Suppose the correct model would use growth rates

$$\Delta y_{1t} = \zeta_1 \Delta y_{1,t-1} + \cdots + \zeta_p \Delta y_{1,t-p} + \boldsymbol{\beta}' \mathbf{x}_{t-1} + \varepsilon_{1t}$$

$$y_{1t} - y_{1,t-1} = \zeta_1 (y_{1,t-1} - y_{1,t-2}) + \cdots + \zeta_p (y_{1,t-p} - y_{1,t-p-1}) + \boldsymbol{\beta}' \mathbf{x}_{t-1} + \varepsilon_{1t}$$

This is a special case of regression in levels

$$y_{1t} = \phi_1 y_{1,t-1} + \cdots + \phi_{p+1} y_{t-p-1} + \boldsymbol{\beta}' \mathbf{x}_{t-1} + \varepsilon_{1t}$$

$$\phi_1 = 1 + \zeta_1 \qquad \phi_2 = \zeta_2 - \zeta_1 \qquad \phi_3 = \zeta_3 - \zeta_2$$

$$\cdots \qquad \phi_p = \zeta_p - \zeta_{p-1} \qquad \phi_{p+1} = -\zeta_p$$

Growth rate regression			Levels regression		predicted
GDPCH{1}	0.14089		GDPLOG{1}	1.125783	1.14089
GDPCH{2}	0.171458		GDPLOG{2}	0.021619	0.030568
GDPCH{3}	0.019889		GDPLOG{3}	-0.14394	-0.15157
GDPCH{4}	0.027745		GDPLOG{4}	0.000463	0.007856
INFLATION{1}	-0.1362		INFLATION{1}	-0.1873	-0.1362
INFLATION{2}	0.109624		INFLATION{2}	0.101033	0.109624
INFLATION{3}	0.019772		INFLATION{3}	0.023217	0.019772
INFLATION{4}	-0.00278		INFLATION{4}	0.033663	-0.00278
FEDFUNDS{1}	0.073564		FEDFUNDS{1}	0.091929	0.073564
FEDFUNDS{2}	-1.51527		FEDFUNDS{2}	-1.53255	-1.51527
FEDFUNDS{3}	1.135869		FEDFUNDS{3}	1.127251	1.135869
FEDFUNDS{4}	-0.00298		FEDFUNDS{4}	-0.11631	-0.00298
Constant	4.458802		Constant	-7.76291	

$$y_{1t} = \phi_1 y_{1,t-1} + \cdots + \phi_{p+1} y_{t-p-1} + \boldsymbol{\beta}' \mathbf{x}_{t-1} + \varepsilon_{1t}$$

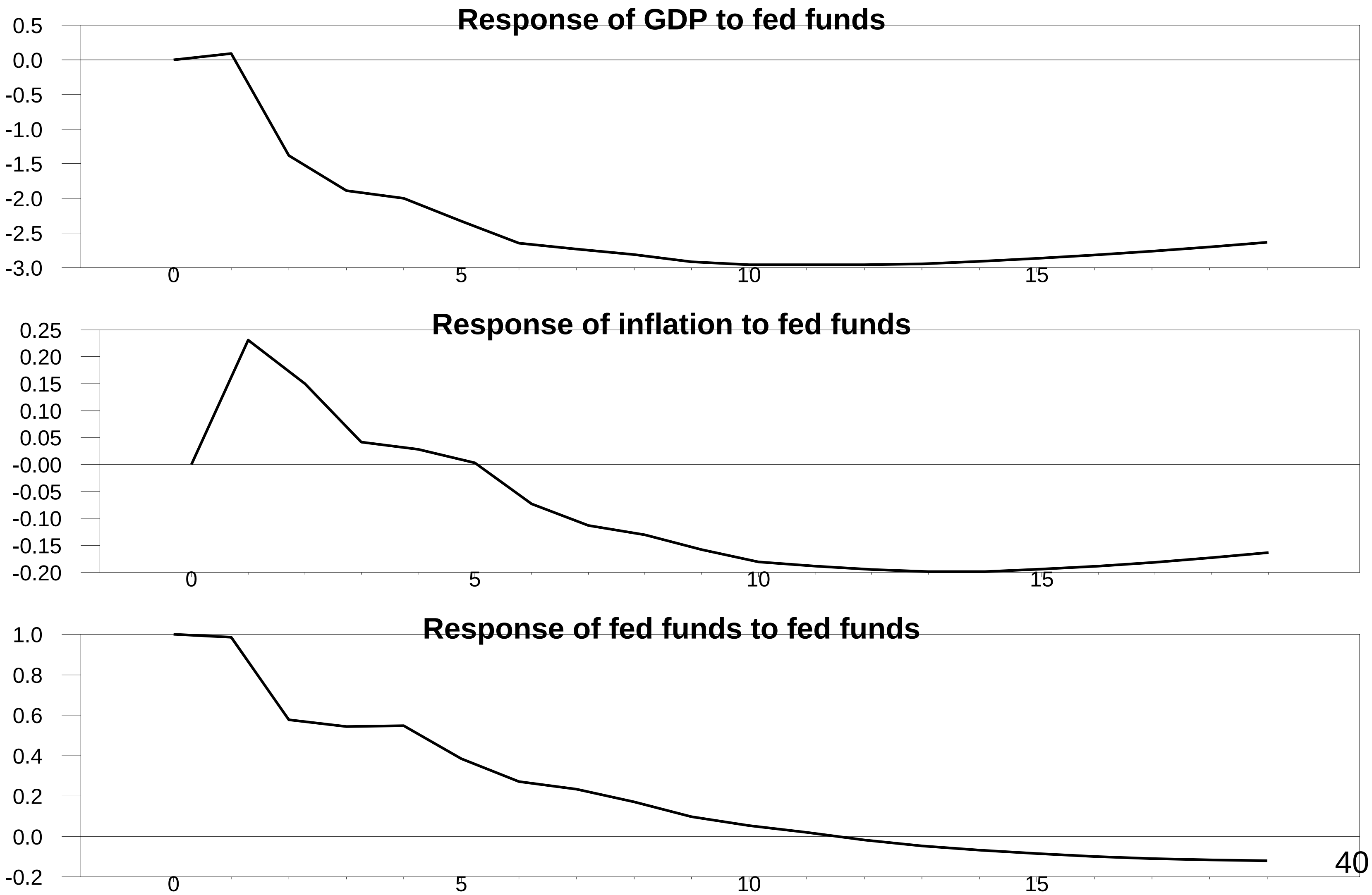
OLS minimizes $T^{-1} \sum_{t=1}^{T} \varepsilon_{1t}^2$

If y_{1t} has a unit root, this will be infinite unless we pick ϕ_j consistent with the growth-rate specification.

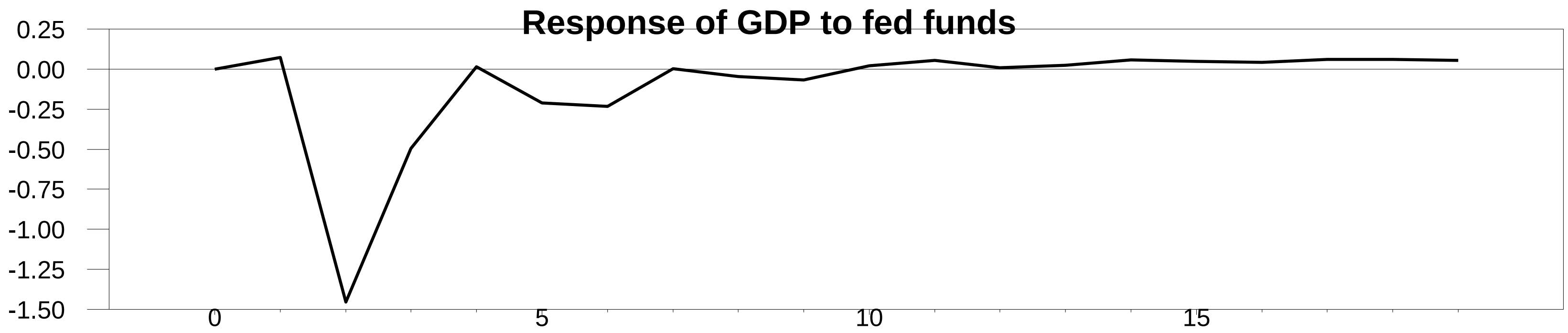
In other words, OLS should force $\hat{\phi}_1 + \hat{\phi}_2 + \hat{\phi}_3 + \hat{\phi}_4$ close to one.

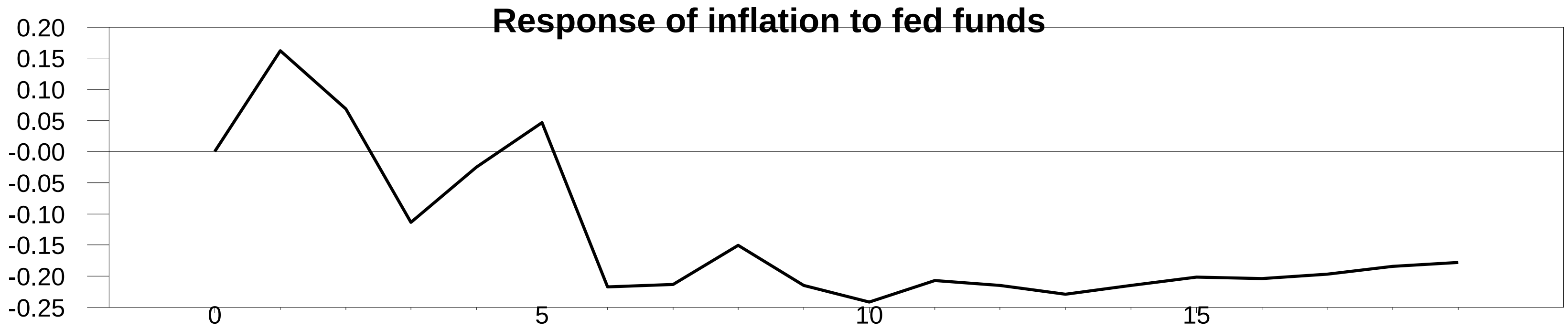
Actual OLS estimate of the sum is 1.004.

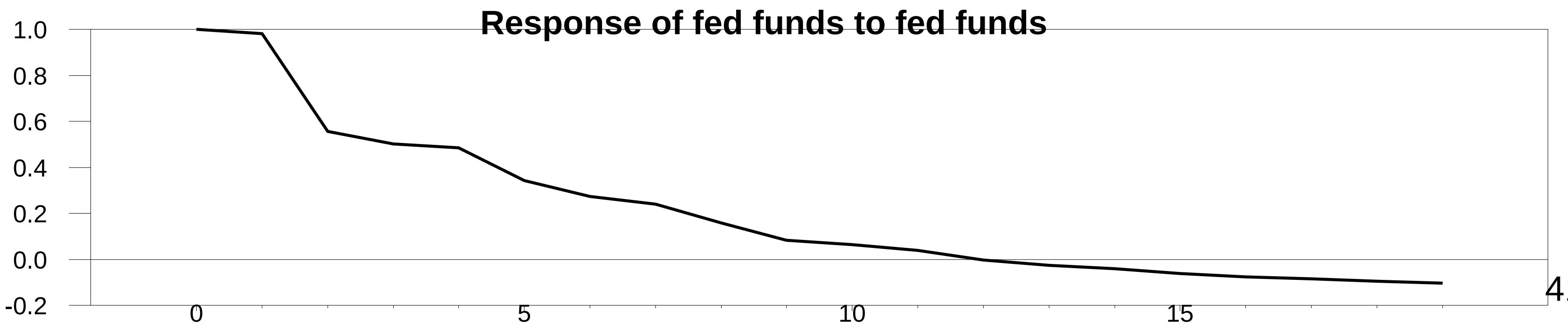
IRF for levels specification



IRF for growth-rate specification







- For this example, levels regression and growth-rate regression are basically estimating the identical system
- If truth is growth-rate, when we estimate in levels we will force OLS to estimate the unit root for us
- But will have more efficient estimates if impose the unit root
- Also can avoid nonstandard distributions for hypothesis tests by using growth rates

- Potential drawbacks to using growth rates
  - GDP may not really have a unit root
  - GDP and price level may be cointegrated
- For this example, baseline specification seems sensible (growth rate of GDP, inflation rate, level of fed funds)

Other implications

- Does not usually make sense to throw in time trend (or quadratic time trend!) in levels regression because growth rates have no trend.
- A simple linear regression of level of a scalar on its own lagged levels is a robust, assumption-free way to remove unknown trend (much better than Hodrick-Prescott filter!)

Proposition: if $\Delta^d y_t$ is stationary for some d, then can write y_{t+h} as a linear function of $y_t, y_{t-1}, \ldots, y_{t-d+1}$ plus a stationary residual.

Example: $d = 1$

$$u_t = \Delta y_t \sim I(0)$$

$$y_{t+h} = y_t + u_{t+1} + u_{t+2} + \cdots + u_{t+h} = y_t + w_t^{(h)}$$

$$w_t^{(h)} = u_{t+1} + u_{t+2} + \cdots + u_{t+h} \sim I(0)$$

Example: $d = 2$

$$u_t = \Delta^2 y_t \sim I(0)$$

$$y_{t+h} = (h+1)y_t - hy_{t-1} + u_{t+h} + 2u_{t+h-1} + \cdots + hu_{t+1}$$

$$= (h+1)y_t - hy_{t-1} + w_t^{(h)}$$

$$w_t^{(h)} = u_{t+h} + 2u_{t+h-1} + \cdots + hu_{t+1} \sim I(0)$$

If $y_t \sim I(2)$, what happens if we regress y_{t+h} on $(1, y_t, y_{t-1})'$?

- If coefficient on $y_t = h + 1$ and coefficient on $y_{t-1} = -h$, then average squared residual will tend to a finite number.
- For any other coefficients, average squared residual will tend to an infinite number.
- OLS will give a consistent estimate of parameters that characterize the trend.

If $y_t \sim I(2)$, what happens if we regress y_{t+h} on $(1, y_t, y_{t-1}, y_{t-2}, y_{t-3})'$?

- Two of the coefficients will make the residuals stationary.
- Other two coefficients will then try to forecast stationary component.

Conclusion: we don't need to know *d*.

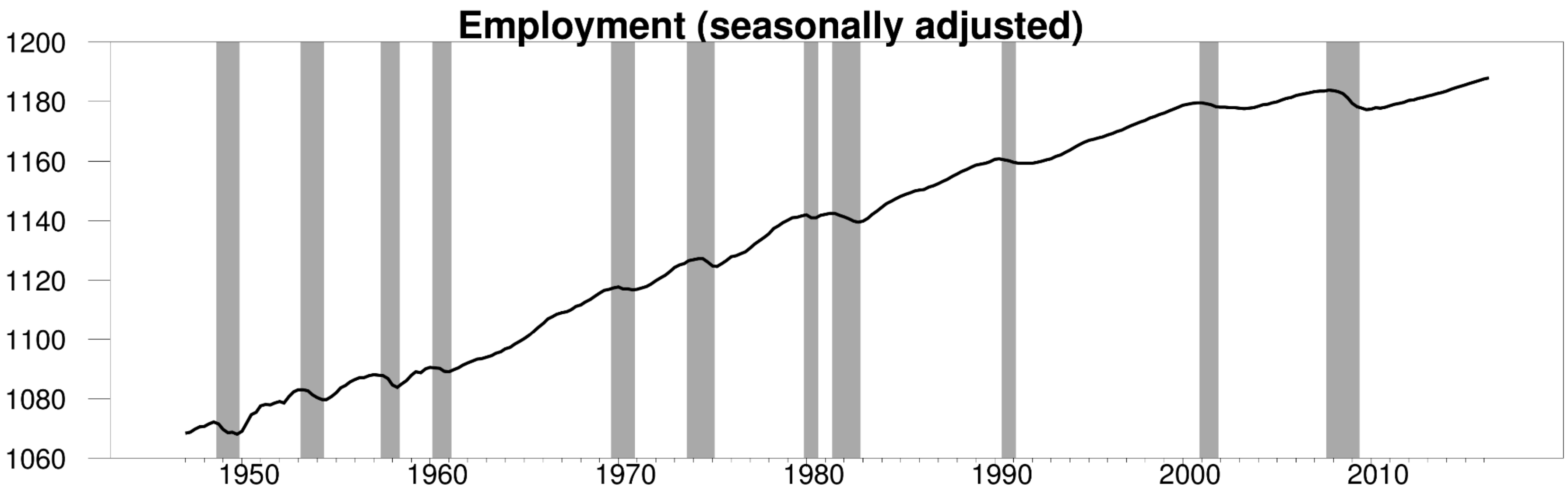
If $y_t \sim I(d)$ for some unknown $d \leq 4$, the population linear projection of y_{t+h} on $(1, y_t, y_{t-1}, y_{t-2}, y_{t-3})'$ exists and can be consistently estimated by OLS regression.

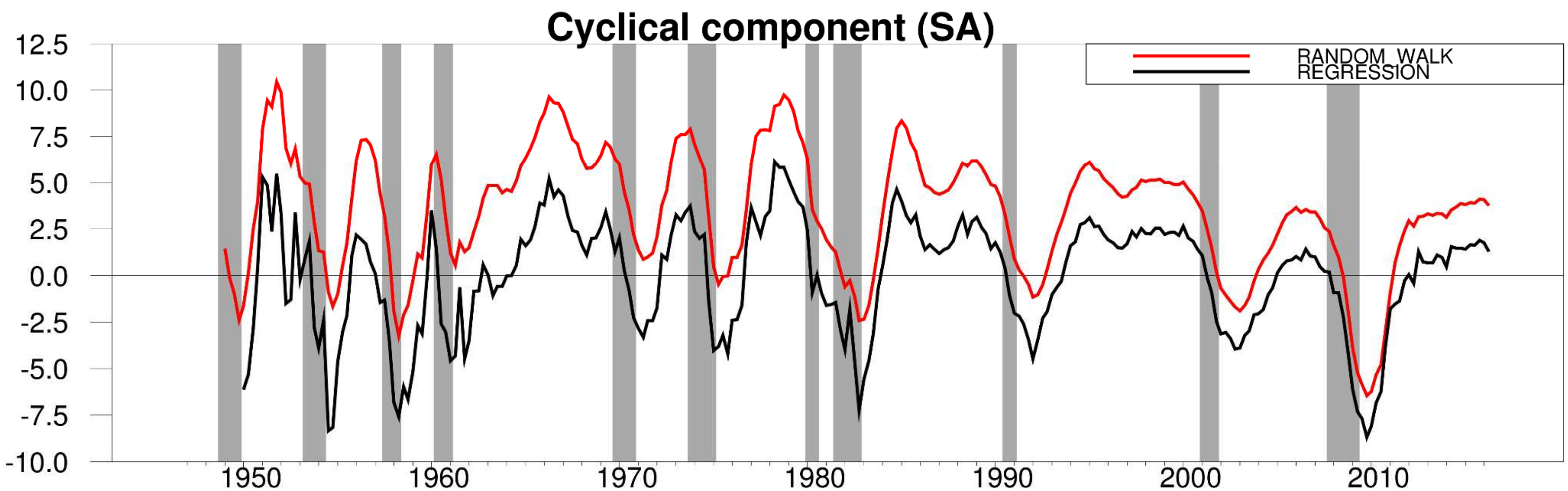
Proposed definition: the cyclical component of y_t is part we can't predict 2 years ahead using linear regression.

For quarterly data estimate by OLS

$$y_{t+8} = \beta_0 + \beta_1 y_t + \beta_2 y_{t-1} + \beta_3 y_{t-2} + \beta_4 y_{t-3} + v_{t+8}$$

interpret the residuals $\hat{v}_{t+8}$ as the cyclical component

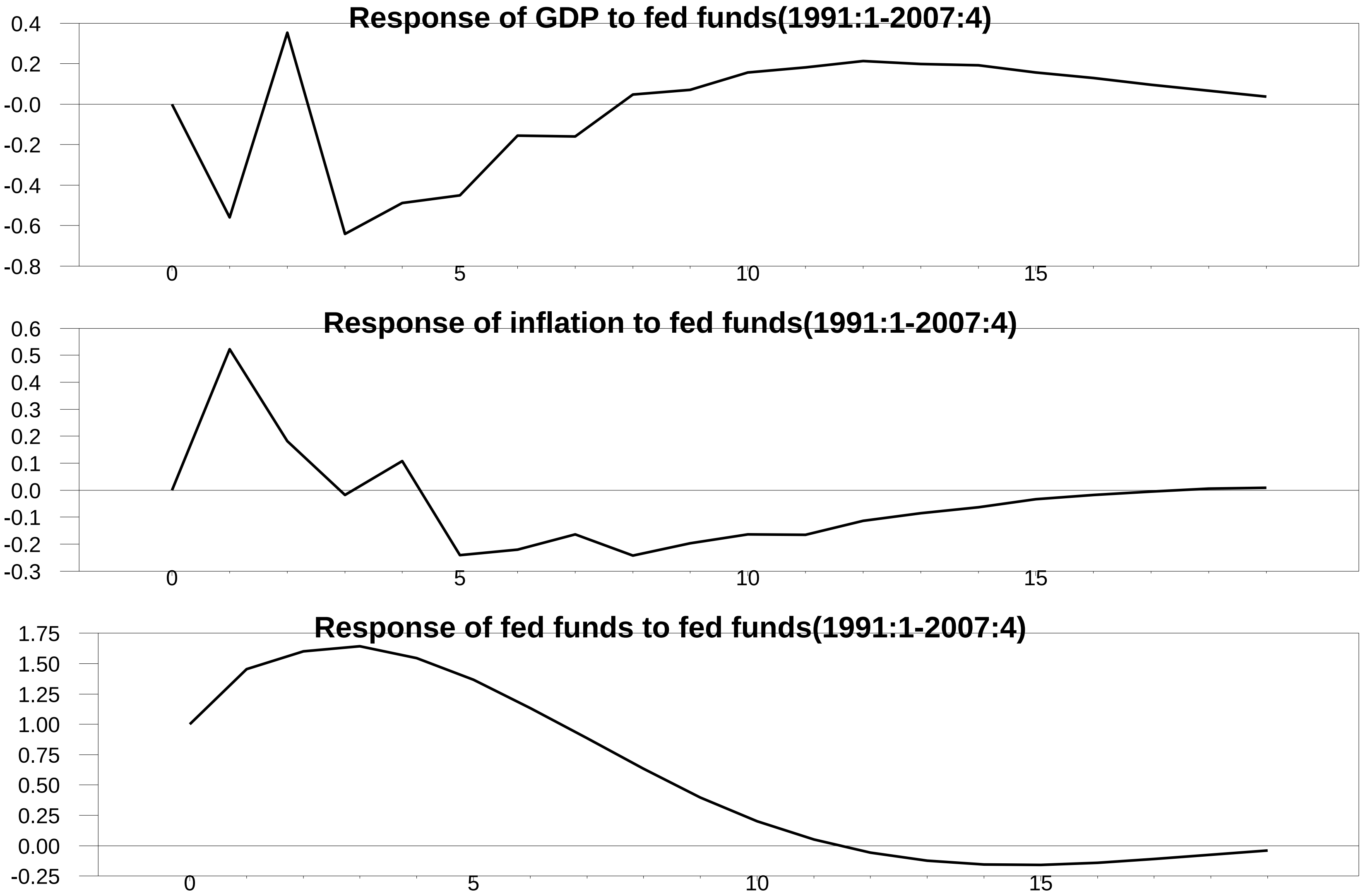

Employment (seasonally adjusted)
1200
1180
1160
1140
1120
1100
1080
1060
1950
1960
1970
1980
1990
2000
2010

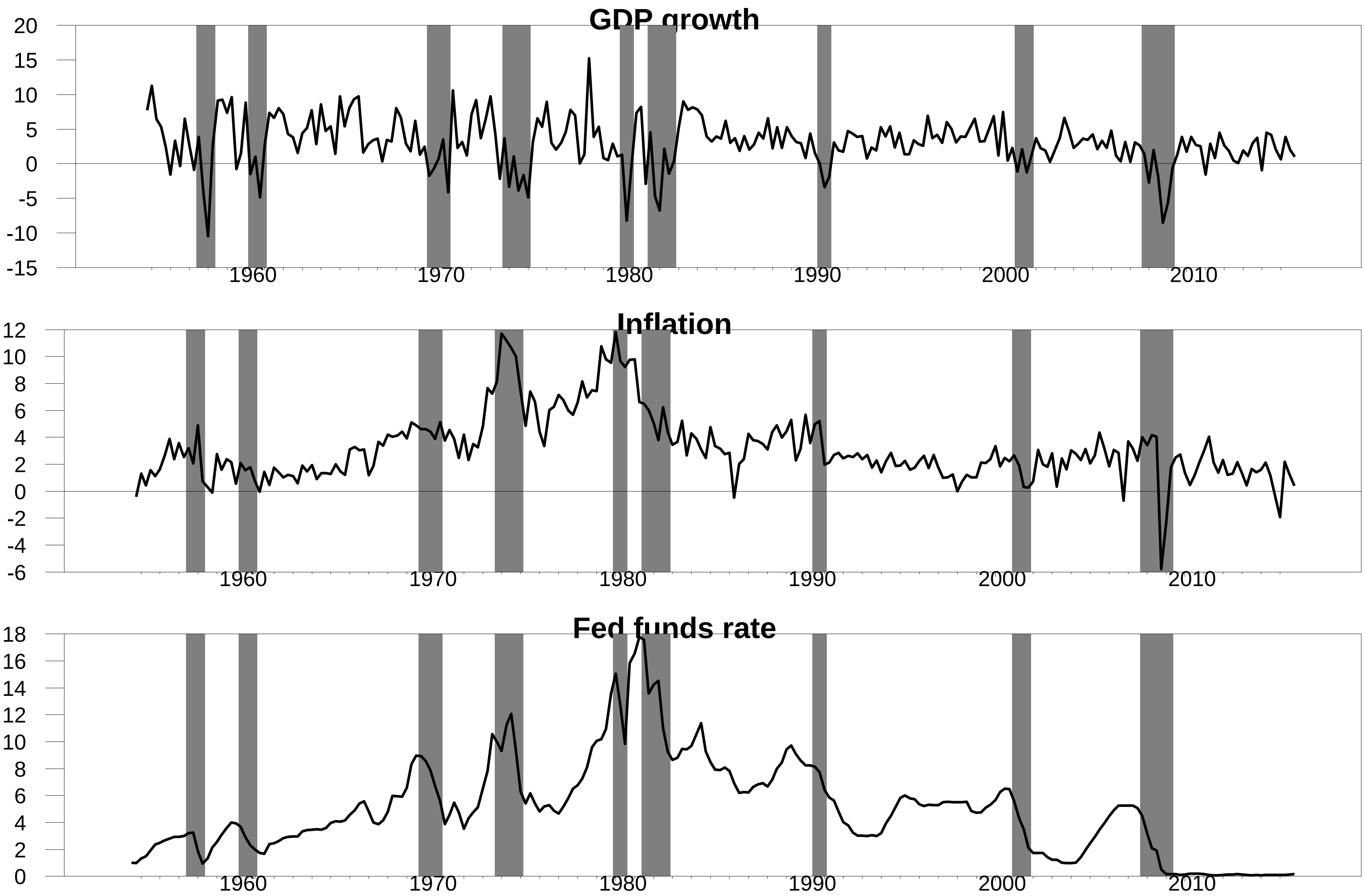

Cyclical component (SA)
RANDOM_WALK
REGRESSION
12.5
10.0
7.5
5.0
2.5
0.0
-2.5
-5.0
-7.5
-10.0
1950
1960
1970
1980
1990
2000
2010

F. Instability

- What happens if we estimate VAR over 1991:Q1 to 2007:Q4?

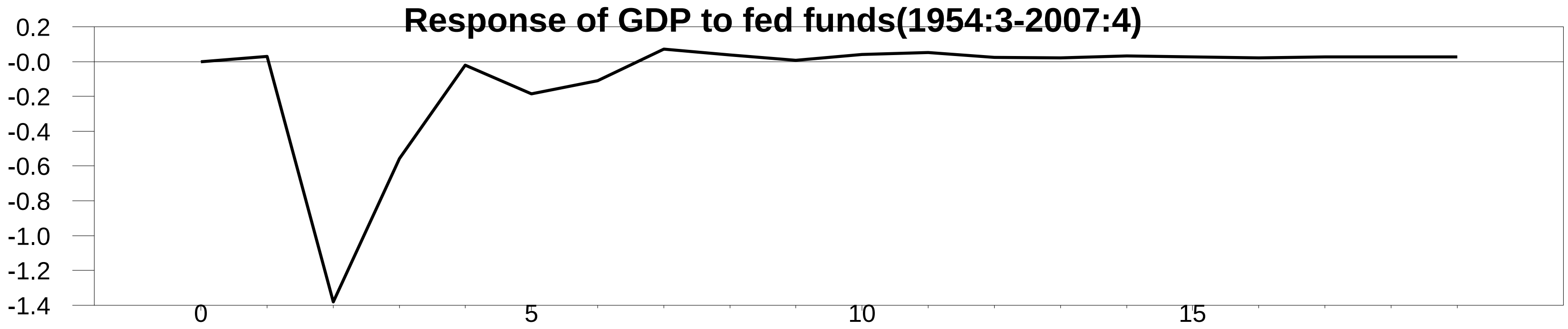
GDP coeffs for 1991:Q1 to 2007:Q4 sample					GDP coeffs for 1960:Q1 to 1990:Q4 sample			
	coeff	std error	t stat			coeff	std error	t stat
GDPCH{1}	0.155077	0.143744	1.07884		GDPCH{1}	0.14089	0.09694	1.45338
GDPCH{2}	0.185423	0.149189	1.24288		GDPCH{2}	0.171458	0.095324	1.79869
GDPCH{3}	-0.10024	0.145688	-0.68807		GDPCH{3}	0.019889	0.095968	0.20725
GDPCH{4}	0.126178	0.145687	0.86609		GDPCH{4}	0.027745	0.088859	0.31223
INFLATION{1}	-0.05773	0.295557	-0.19531		INFLATION{1}	-0.1362	0.255978	-0.53208
INFLATION{2}	-0.16592	0.274807	-0.60378		INFLATION{2}	0.109624	0.293745	0.37319
INFLATION{3}	-0.30131	0.27268	-1.10499		INFLATION{3}	0.019772	0.294628	0.06711
INFLATION{4}	0.057866	0.278822	0.20754		INFLATION{4}	-0.00278	0.265583	-0.01046
FEDFUNDS{1}	-0.55891	0.948064	-0.58952		FEDFUNDS{1}	0.073564	0.324268	0.22686
FEDFUNDS{2}	1.282767	1.700743	0.75424		FEDFUNDS{2}	-1.51527	0.434676	-3.48598
FEDFUNDS{3}	-1.46413	1.667997	-0.87778		FEDFUNDS{3}	1.135869	0.459483	2.47206
FEDFUNDS{4}	0.670807	0.868767	0.77214		FEDFUNDS{4}	-0.00298	0.340934	-0.00873
Constant	3.253064	1.496104	2.17436		Constant	4.458802	1.322775	3.37079

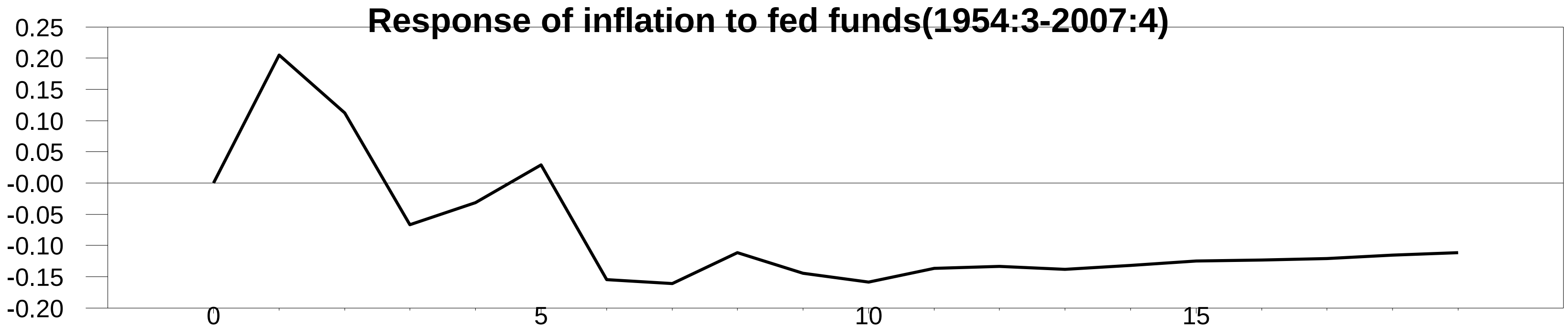

Response of GDP to fed funds(1991:1-2007:4)
0.4
0.2
-0.0
-0.2
-0.4
-0.6
-0.8
0
5
10
15
Response of inflation to fed funds(1991:1-2007:4)
0.6
0.5
0.4
0.3
0.2
0.1
0.0
-0.1
-0.2
-0.3
0
5
10
15
Response of fed funds to fed funds(1991:1-2007:4)
1.75
1.50
1.25
1.00
0.75
0.50
0.25
0.00
-0.25
0
5
10
15

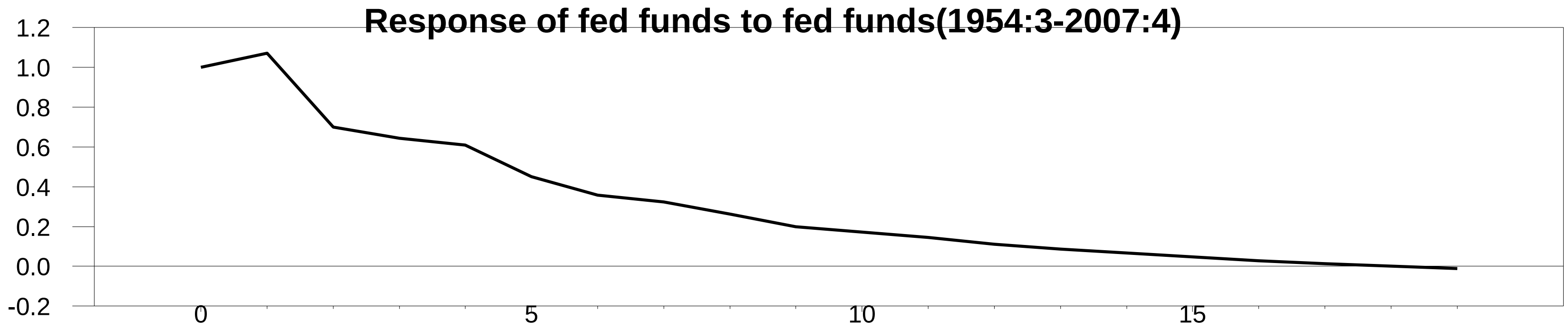

GDP growth
20
15
10
5
0
-5
-10
-15
1960
1970
1980
1990
2000
2010
Inflation
12
10
8
6
4
2
0
-2
-4
-6
1960
1970
1980
1990
2000
2010
Fed funds rate
18
16
14
12
10
8
6
4
2
0
1960
1970
1980
1990
2000
2010

Options for dealing with instability

- Estimate allowing for GARCH to reduce impact of outliers (Hamilton, 2010)
- Find generalization of model that is stable
- Use Bayesian methods to bring in additional information
- Estimate system with time-varying parameters or changes in regime
- Use full sample as average summary (plim of regression)


Response of GDP to fed funds(1954:3-2007:4)
0.2
-0.0
-0.2
-0.4
-0.6
-0.8
-1.0
-1.2
-1.4
0
5
10
15


Response of inflation to fed funds(1954:3-2007:4)
0.25
0.20
0.15
0.10
0.05
-0.00
-0.05
-0.10
-0.15
-0.20
0
5
10
15


Response of fed funds to fed funds(1954:3-2007:4)
1.2
1.0
0.8
0.6
0.4
0.2
0.0
-0.2
0
5
10
15