Answer key for the midterm in 2020

1a.) $b = T^{-1}\sum_{t=1}^{T} y_t = \bar{y}$ $\quad s^2 = (T-1)^{-1}\sum_{t=1}^{T}(y_t - \bar{y})^2$

b.)

i.) $\bar{y}$ has mean β, variance σ^2/T and is linear combination of multivariate Normal. Hence $\bar{y} \sim N(\beta, \sigma^2/T)$

ii.) From (i), $\sqrt{T}\bar{y}/\sigma \sim N(\beta\sqrt{T}/\sigma, 1)$. When $\beta = 0$, square of this is $\chi^2(1)$ so $(\bar{y}^2/\sigma^2)$ is $(1/T)$ times $\chi^2(1)$ when $\beta = 0$.

iii.) $\sum_{t=1}^{T}(y_t - \bar{y})^2/\sigma^2 = \mathbf{v}'\mathbf{M}\mathbf{v}$ for $\mathbf{M} = (\mathbf{I}_T - \mathbf{1}(\mathbf{1}'\mathbf{1})^{-1}\mathbf{1}') = \mathbf{Q}\boldsymbol{\Lambda}\mathbf{Q}'$ for $\mathbf{Q}$ orthonormal and $\boldsymbol{\Lambda}$ is diagonal with $T-1$ ones and one zero on principal diagonal and $\mathbf{v} = \mathbf{y}/\sigma \sim N(\mathbf{0}, \mathbf{I}_T)$. Hence $\mathbf{h} = \mathbf{Q}'\mathbf{v} \sim N(\mathbf{0}, \mathbf{I}_T)$ and $\sum_{t=1}^{T}(y_t - \bar{y})^2/\sigma^2 = \sum_{t=1}^{T-1} h_t^2$ is the sum of squares of $(T-1)$ independent $\chi^2(1)$ which sum is $\chi^2(T-1)$.

c.) $SSR_R = \sum_{t=1}^{T} y_t^2$ $\quad SSR_U = \sum_{t=1}^{T}(y_t - \bar{y})^2$

$$SSR_R - SSR_U = \sum_{t=1}^{T} y_t^2 - \left[\sum_{t=1}^{T} y_t^2 - 2T\bar{y}^2 + T\bar{y}^2\right] = T\bar{y}^2$$

$$H = \frac{m^{-1}(SSR_R - SSR_U)}{SSR_U/(T-k)} = \frac{T\bar{y}^2}{\sum_{t=1}^{T}(y_t-\bar{y})^2/(T-1)}$$

d.) $H = \dfrac{T\bar{y}^2/\sigma^2}{\sum_{t=1}^{T}(y_t-\bar{y})^2/[(T-1)\sigma^2]}$

From (b), numerator is $\chi^2(1)$ and denominator is $\chi^2(T-1)$. Moreover, the $(T \times 1)$ vector whose tth element is $y_t - \bar{y}$ is given by $\mathbf{My}$ which has covariance with $\bar{y} = T^{-1}\mathbf{y}'\mathbf{1}$ given by $E(\mathbf{M}\mathbf{y}\mathbf{y}'\mathbf{1}) = \mathbf{M}\sigma^2\mathbf{I}_T\mathbf{1} = \mathbf{0}$. Since $\mathbf{My}$ is uncorrelated with $\mathbf{y}'\mathbf{1}$ and since multivariate Normal, the numerator and denominator are independent. Hence $H \sim F(1, T-1)$.

e.) $H = \dfrac{(\sqrt{T}\bar{y})^2}{\sum_{t=1}^{T}(y_t-\bar{y})^2/(T-1)}$

$\sqrt{T}\bar{y} \xrightarrow{L} N(0, \sigma^2)$ by CLT so $(\sqrt{T}\bar{y})^2 \xrightarrow{L} \sigma^2\chi^1(1)$. Also

$\dfrac{\sum_{t=1}^{T}(y_t-\bar{y})^2}{T-1} = (T-1)^{-1}\sum_{t=1}^{T} y_t^2 - \frac{T}{T-1}\bar{y}^2 \xrightarrow{p} \sigma^2$ by LLN. Hence $H \xrightarrow{L} \chi^2(1)$. No other assumptions needed.

f.) $\tau = \dfrac{b}{\sqrt{s^2/T}} = \dfrac{\sqrt{T}\bar{y}}{\sqrt{\sum_{t=1}^{T}(y_t-\bar{y})^2/(T-1)}} = \sqrt{H} \times \text{sign}(\bar{y})$

2a.) $\hat{\boldsymbol{\beta}}_{GLS} = (\mathbf{X}'\mathbf{V}^{-1}\mathbf{X})^{-1}(\mathbf{X}'\mathbf{V}^{-1}\mathbf{y})$ for $\mathbf{V} = \begin{bmatrix} \mathbf{I}_{T_1} & \mathbf{0} \\ \mathbf{0} & \lambda\mathbf{I}_{T_2} \end{bmatrix}$

$$= \left\{\begin{bmatrix} \mathbf{X}_1' & \mathbf{X}_2' \end{bmatrix}\begin{bmatrix} \mathbf{I}_{T_1} & \mathbf{0} \\ \mathbf{0} & \lambda\mathbf{I}_{T_2} \end{bmatrix}^{-1}\begin{bmatrix} \mathbf{X}_1 \\ \mathbf{X}_2 \end{bmatrix}\right\}^{-1}\left\{\begin{bmatrix} \mathbf{X}_1' & \mathbf{X}_2' \end{bmatrix}\begin{bmatrix} \mathbf{I}_{T_1} & \mathbf{0} \\ \mathbf{0} & \lambda\mathbf{I}_{T_2} \end{bmatrix}^{-1}\begin{bmatrix} \mathbf{y}_1 \\ \mathbf{y}_2 \end{bmatrix}\right\}$$

$= (\mathbf{X}_1'\mathbf{X}_1 + \lambda^{-1}\mathbf{X}_2'\mathbf{X}_2)^{-1}(\mathbf{X}_1'\mathbf{y}_1 + \lambda^{-1}\mathbf{X}_2'\mathbf{y}_2)$.

This compares with $\mathbf{b}_{OLS} = (\mathbf{X}_1'\mathbf{X}_1 + \mathbf{X}_2'\mathbf{X}_2)^{-1}(\mathbf{X}_1'\mathbf{y}_1 + \mathbf{X}_2'\mathbf{y}_2)$. GLS downweights observations in the second half of the sample relative to OLS because the second-half observations are less reliable.

b.) GLS will have a smaller variance than OLS, that is,

$$E\left[(\mathbf{b}_{OLS} - \boldsymbol{\beta})(\mathbf{b}_{OLS} - \boldsymbol{\beta})'|\mathbf{X}\right] - E\left[\left(\hat{\boldsymbol{\beta}}_{GLS} - \boldsymbol{\beta}\right)\left(\hat{\boldsymbol{\beta}}_{GLS} - \boldsymbol{\beta}\right)'|\mathbf{X}\right]$$

is a positive semidefinite matrix. In other words, GLS gives a more accurate estimate than OLS. Also, if we estimated by OLS, the OLS standard errors would be inaccurate.

c.) Let $s_1^2 = (T_1 - k)^{-1}\sum_{t=1}^{T_1}(y_t - \mathbf{x}_t'\mathbf{b}_1)^2$ for $\mathbf{b}_1 = (\mathbf{X}_1'\mathbf{X}_1)^{-1}(\mathbf{X}_1'\mathbf{y}_1)$

$s_2^2 = (T_2 - k)^{-1}\sum_{t=T_1+1}^{T}(y_t - \mathbf{x}_t'\mathbf{b}_2)^2$ for $\mathbf{b}_2 = (\mathbf{X}_2'\mathbf{X}_2)^{-1}(\mathbf{X}_2'\mathbf{y}_2)$

Could try $\hat{\lambda} = s_2^2/s_1^2$. A little better estimate is probably

$$\tilde{\lambda} = \frac{T_1^{-1} \sum_{t=1}^{T_1} (y_t - \mathbf{x}_t' \hat{\boldsymbol{\beta}}_{GLS})^2}{T_2^{-1} \sum_{t=T_1+1}^{T} (y_t - \mathbf{x}_t' \hat{\boldsymbol{\beta}}_{GLS})^2}.$$