Econ 220B, Winter 2020

Midterm Exam

DIRECTIONS: No books or notes of any kind are allowed. Answer all questions on separate paper. 150 points are possible on this exam.

1.) (100 points total) Hint: if you can't derive some of the results below, just take a guess at the correct answer to receive partial credit.

An OLS regression model relating a scalar y_t to a $(k \times 1)$ vector $\mathbf{x}_t$ takes the form $y_t = \mathbf{x}_t'\boldsymbol{\beta} + \varepsilon_t$ where $E(\varepsilon_t) = 0$ and $E(\varepsilon_t^2) = \sigma^2$. This question considers the special case of a single explanatory variable $(k = 1)$ consisting of a constant term. That is, $\mathbf{x}_t = 1$ for $t = 1, ..., T$:

$$y_t = \beta + \varepsilon_t.$$

a.) (20 points) Write down the expressions for the OLS estimate b of the scalar β and the OLS estimate s^2 of the scalar σ^2 for this special case when $k = 1$.

b.) (30 points)Suppose that $\varepsilon_t \sim$ i.i.d. $N(0, \sigma^2)$. Under this assumption, calculate the exact small-sample distribution of

i.) $\bar{y} = T^{-1} \sum_{t=1}^{T} y_t$

ii.) $\bar{y}^2/\sigma^2$

iii.) $\sum_{t=1}^{T} (y_t - \bar{y})^2/\sigma^2$.

c.) (10 points) In the general regression model with $k > 1$, one way to test the null hypothesis $H_0 : \mathbf{R}\boldsymbol{\beta} = \mathbf{r}$ where $\mathbf{R}$ is an $(m \times k)$ matrix is: (1) calculate the sum of squared residuals SSR_U when no restrictions are imposed, and (2) compare this with the sum of squared residuals SSR_R when H_0 is imposed by looking at the statistic

$$H = \frac{m^{-1}(SSR_R - SSR_U)}{SSR_U/(T - k)}.$$

Calculate the value of H for testing the null hypothesis $H_0 : \beta = 0$ in the special case of $\mathbf{x}_t = 1$. Calculate the expression for H for this special case and simplify as much as possible.

d.) (10 points) Use your results from part (b) and (c) to calculate the exact small-sample distribution of H for this special case when $k = 1$. Please do not repeat or appeal to any general results or derivations you may remember for a more general case when $k > 1$, but instead derive the distribution for the case $k = 1$ from first principles directly from the expressions you obtained in part (b) and (c).

e.) (15 points) Now replace the assumption that $\varepsilon_t \sim$ i.i.d. $N(0, \sigma^2)$ with the assumption that ε_t is a stationary ergodic martingale difference sequence with variance σ^2. Calculate the asymptotic distribution of your expression for H in part (c). Again do not use any more general results or derivations but derive the distribution directly from first principles from the expression you obtained in part (c). Do you need any additional assumptions in addition to those stated here in this question in order to obtain this result?

f.) (15 points) Another way to test the hypothesis $H_0 : \beta = 0$ is with the OLS t-test of this hypothesis. Write down the formula for the OLS t-test, again focusing only on the special case when $k = 1$ (only partial credit will be given for writing down an expression that would be appropriate when $k > 1$). Can you show the exact mathematical relation between your expressions in part (c) and part (f)?

2.) (50 points total) Consider an OLS regression model relating a scalar y_t to a $(k \times 1)$ vector $\mathbf{x}_t$ of the form $y_t = \mathbf{x}_t'\boldsymbol{\beta} + \varepsilon_t$. Suppose we have a sample of size $T = T_1 + T_2$ where we believe that the residuals for the first T_1 observations have variance σ^2 and the residuals for the last T_2 observations have variance $\lambda\sigma^2$ for $\lambda > 1$. The regression model can be written

$$\begin{bmatrix} \begin{bmatrix} y_1 \\ \vdots \\ y_{T_1} \end{bmatrix} \\ \begin{bmatrix} y_{T_1+1} \\ \vdots \\ y_{T_1+T_2} \end{bmatrix} \end{bmatrix} = \begin{bmatrix} \begin{bmatrix} \mathbf{x}_1' \\ \vdots \\ \mathbf{x}_{T_1}' \end{bmatrix} \\ \begin{bmatrix} \mathbf{x}_{T_1+1}' \\ \vdots \\ \mathbf{x}_{T_1+T_2}' \end{bmatrix} \end{bmatrix} \begin{bmatrix} \beta_1 \\ \vdots \\ \beta_k \end{bmatrix} + \begin{bmatrix} \begin{bmatrix} \varepsilon_1 \\ \vdots \\ \varepsilon_{T_1} \end{bmatrix} \\ \begin{bmatrix} \varepsilon_{T_1+1} \\ \vdots \\ \varepsilon_{T_1+T_2} \end{bmatrix} \end{bmatrix}$$

$$\begin{bmatrix} \underset{(T_1\times 1)}{\mathbf{y}_1} \\ \underset{(T_2\times 1)}{\mathbf{y}_2} \end{bmatrix} = \begin{bmatrix} \underset{(T_1\times k)}{\mathbf{X}_1} \\ \underset{(T_2\times k)}{\mathbf{X}_2} \end{bmatrix} \underset{(k\times 1)}{\boldsymbol{\beta}} + \begin{bmatrix} \underset{(T_1\times 1)}{\boldsymbol{\varepsilon}_1} \\ \underset{(T_2\times 1)}{\boldsymbol{\varepsilon}_2} \end{bmatrix}$$

$$\underset{(T\times 1)}{\mathbf{y}} = \underset{(T\times k)}{\mathbf{X}}\underset{(k\times 1)}{\boldsymbol{\beta}} + \underset{(T\times 1)}{\boldsymbol{\varepsilon}}$$

$$E(\boldsymbol{\varepsilon}|\mathbf{X}) = \mathbf{0}$$

$$E(\boldsymbol{\varepsilon}\boldsymbol{\varepsilon}'|\mathbf{X}) = \begin{bmatrix} \sigma^2\mathbf{I}_{T_1} & \mathbf{0} \\ \mathbf{0} & \lambda\sigma^2\mathbf{I}_{T_2} \end{bmatrix}.$$

a.) (20 points) Calculate the GLS estimate of $\boldsymbol{\beta}$ for this particular case. Simplify the expression as much as possible so that you can explain intuitively what GLS amounts to in this case in terms of the expression you derive.

b.) (20 points) What are the major benefits of GLS compared to OLS for this particular case?

c.) (10 points) Can you suggest (based on your intuition from part (a)) an estimate you might use for λ?