Answer key for the midterm in 2018

1a.)

$$\mathbf{b} = \begin{bmatrix} \sum x_{t1}^2 & 0 \\ 0 & \sum x_{t2}^2 \end{bmatrix}^{-1} \begin{bmatrix} \sum x_{t1}y_t \\ \sum x_{t2}y_t \end{bmatrix}$$

$$b_1 = \frac{\sum x_{t1}y_t}{\sum x_{t1}^2}$$

$$\tau_1 = \frac{b_1\sqrt{\sum x_{t1}^2}}{\sqrt{s^2}}$$

for $s^2 = (T-2)^{-1}\sum(y_t - x_{t1}b_1 - x_{t2}b_2)^2$ and τ_1 distributed Student t with $(T-2)$ d.f.

b.)

$$\tau_2 = \frac{b_2\sqrt{\sum x_{t2}^2}}{\sqrt{s^2}}.$$

c.)

$$F = (1/2)\begin{bmatrix} b_1 & b_2 \end{bmatrix}\left\{ s^2 \begin{bmatrix} \sum x_{t1}^2 & 0 \\ 0 & \sum x_{t2}^2 \end{bmatrix}^{-1}\right\} \begin{bmatrix} b_1 \\ b_2 \end{bmatrix}$$

$$= (1/2)\left(\frac{b_1^2\sum x_{t1}^2}{s^2} + \frac{b_2^2\sum x_{t2}^2}{s^2}\right)$$

$$= (1/2)(\tau_1^2 + \tau_2^2) \sim F(2, T-2)$$

d.)

$$\tilde{b}_1 = \frac{\sum x_{t1}y_t}{\sum x_{t1}^2} = b_1.$$

Estimates identical because could calculate b_1 by regressing y_t on residuals from a first-stage regression of x_{t1} on x_{t2}. But since x_{t1} is orthogonal to x_{t2}, the residuals from that first-stage regression are numerically identical to x_{t1} itself.

2a.) $\hat{h} = \mathbf{c}'\mathbf{b}$ for $\mathbf{b} = (\sum \mathbf{x}_t\mathbf{x}_t')^{-1}\sum \mathbf{x}_t y_t$ and $\mathbf{c}' = \begin{bmatrix} 1 & 1 & 0 & \cdots & 0 \end{bmatrix}$

$\sqrt{T}(\mathbf{b} - \boldsymbol{\beta}) = (T^{-1}\sum \mathbf{x}_t\mathbf{x}_t')^{-1} T^{-1/2}\sum \mathbf{x}_t\varepsilon_t \xrightarrow{L} \mathbf{Q}^{-1}\mathbf{v}$ for $\mathbf{v} \sim N(\mathbf{0}, \mathbf{S})$ so

$\sqrt{T}(\mathbf{b} - \boldsymbol{\beta}) \xrightarrow{L} N(\mathbf{0}, \mathbf{Q}^{-1}\mathbf{S}\mathbf{Q}^{-1})$ and $\sqrt{T}(\hat{h} - 1) \xrightarrow{L} N(\mathbf{0}, \mathbf{c}'\mathbf{Q}^{-1}\mathbf{S}\mathbf{Q}^{-1}\mathbf{c})$

b.) Let $\hat{\mathbf{V}} = \hat{\mathbf{Q}}^{-1}\hat{\mathbf{S}}\hat{\mathbf{Q}}^{-1}$ for $\hat{\mathbf{Q}} = T^{-1}\sum \mathbf{x}_t\mathbf{x}_t'$ and $\hat{\mathbf{S}} = T^{-1}\sum e_t^2\mathbf{x}_t\mathbf{x}_t'$ and $e_t = y_t - \mathbf{x}_t'\mathbf{b}$. Then test statistic is

$$\frac{\sqrt{T}(\hat{h} - 1)}{\sqrt{\mathbf{c}'\hat{\mathbf{V}}\mathbf{c}}} \xrightarrow{L} N(0,1).$$

c.) Would still be valid, but usual OLS t-statistic is also valid and likely closer to $N(0,1)$

$$\frac{\sqrt{T}(\hat{h} - 1)}{\sqrt{s^2\mathbf{c}'\hat{\mathbf{Q}}^{-1}\mathbf{c}}}$$

for $s^2 = (T-k)^{-1}\sum e_t^2$ because σ^2 easier to estimate than $\mathbf{S}$.