Answer key for the midterm in 2016

1a.) classical regression model

b.) $\mathbf{b} = (\mathbf{X}'\mathbf{X})^{-1}\mathbf{X}'\mathbf{y} = \boldsymbol{\beta} + (\mathbf{X}'\mathbf{X})^{-1}\mathbf{X}'\boldsymbol{\varepsilon}$. So $E(\mathbf{b}|\mathbf{X}) = \boldsymbol{\beta}$ and $E(\mathbf{b}) = \boldsymbol{\beta}$. This means that $\mathbf{b}$ is an unbiased estimate of $\boldsymbol{\beta}$.

c.) The (1,1) element is the conditional variance of the estimate of the first element of $\boldsymbol{\beta}$.

$$\begin{aligned} E[(\mathbf{b}-\boldsymbol{\beta})(\mathbf{b}-\boldsymbol{\beta})'|\mathbf{X}] &= E[(\mathbf{X}'\mathbf{X})^{-1}\mathbf{X}'\boldsymbol{\varepsilon}\boldsymbol{\varepsilon}'\mathbf{X}(\mathbf{X}'\mathbf{X})^{-1}|\mathbf{X}] \\ &= (\mathbf{X}'\mathbf{X})^{-1}\mathbf{X}'(\lambda^2\mathbf{X}\mathbf{X}' + \sigma^2\mathbf{I}_T)\mathbf{X}(\mathbf{X}'\mathbf{X})^{-1} \\ &= \lambda^2\mathbf{I}_k + \sigma^2(\mathbf{X}'\mathbf{X})^{-1}. \end{aligned}$$

This reduces to the classical regression result $\sigma^2(\mathbf{X}'\mathbf{X})^{-1}$ when $\lambda = 0$.

d.)

$$\begin{aligned} E(s^2|\mathbf{X}) &= (T-k)^{-1}E(\boldsymbol{\varepsilon}'\mathbf{M}\boldsymbol{\varepsilon}) \\ &= (T-k)^{-1}\text{trace}[E(\mathbf{M}\boldsymbol{\varepsilon}\boldsymbol{\varepsilon}'|\mathbf{X}] \\ &= (T-k)^{-1}\text{trace}[\mathbf{M}(\lambda^2\mathbf{X}\mathbf{X}' + \sigma^2\mathbf{I}_T)] \\ &= (T-k)^{-1}\text{trace}[\sigma^2\mathbf{M}] \\ &= \sigma^2. \end{aligned}$$

2a.) Assumptions imply that ε_t^2 is i.i.d. with mean σ^2 and variance $E(\varepsilon_t^2 - \sigma^2)^2 = E(\varepsilon_t^4) - \sigma^4 = 2\sigma^4$ so $\sqrt{T}\left(T^{-1}\sum_{t=1}^T \varepsilon_t^2 - \sigma^2\right) \xrightarrow{L} N(0, 2\sigma^4)$ by Lindeburg-Levy CLT.

b.) This follows immediately from the Ergodic Theorem with $\mathbf{Q} = E(\mathbf{x}_t\mathbf{x}_t')$.

c.) $\sqrt{T}(\mathbf{b}-\boldsymbol{\beta}) = \left(T^{-1}\sum_{t=1}^T \mathbf{x}_t\mathbf{x}_t'\right)^{-1}\left(T^{-1/2}\sum_{t=1}^T \mathbf{x}_t\varepsilon_t\right)$. The given conditions imply that

$$E(\mathbf{x}_t\varepsilon_t|\mathbf{x}_{t-1}\varepsilon_{t-1},\ldots,\mathbf{x}_1\varepsilon_1) = E(\varepsilon_t|\mathbf{x}_{t-1}\varepsilon_{t-1},\ldots,\mathbf{x}_1\varepsilon_1)E(\mathbf{x}_t|\mathbf{x}_{t-1}\varepsilon_{t-1},\ldots,\mathbf{x}_1\varepsilon_1) = \mathbf{0}.$$

So $\mathbf{x}_t\varepsilon_t$ is a martingale difference sequence with variance $\sigma^2\mathbf{Q}$. Hence $T^{-1/2}\sum_{t=1}^T \mathbf{x}_t\varepsilon_t \xrightarrow{L} N(\mathbf{0}, \sigma^2\mathbf{Q})$ and $\sqrt{T}(\mathbf{b}-\boldsymbol{\beta}) \xrightarrow{L} N(\mathbf{0}, \mathbf{Q}^{-1}\sigma^2\mathbf{Q}\mathbf{Q}^{-1})$.

d.)

$$\begin{aligned} T^{-1/2}\sum_{t=1}^T(\mathbf{x}_t'\mathbf{b} - \mathbf{x}_t'\boldsymbol{\beta})^2 &= T^{-1/2}(\mathbf{b}-\boldsymbol{\beta})'\sum_{t=1}^T \mathbf{x}_t\mathbf{x}_t'(\mathbf{b}-\beta) \\ &= [T^{1/2}(\mathbf{b}-\boldsymbol{\beta})']\left(T^{-1}\sum_{t=1}^T \mathbf{x}_t\mathbf{x}_t'\right)(\mathbf{b}-\beta). \end{aligned}$$

Result (c) implies that the first term converges in distribution to $N(\mathbf{0}, \sigma^2\mathbf{Q}^{-1})$, result (b) implies that the second term converges in probability to $\mathbf{Q}$, and result (c) implies that the last term converges in probability to zero.

e.)

$$\begin{aligned} \sum_{t=1}^T(y_t - \mathbf{x}_t'\boldsymbol{\beta})^2 &= \sum_{t=1}^T(y_t - \mathbf{x}_t'\mathbf{b} + \mathbf{x}_t'\mathbf{b} - \mathbf{x}_t'\boldsymbol{\beta})^2 \\ & \sum_{t=1}^T(y_t - \mathbf{x}_t'\mathbf{b})^2 + \sum_{t=1}^T(\mathbf{x}_t'\mathbf{b} - \mathbf{x}_t'\boldsymbol{\beta})^2 \end{aligned}$$

where the cross-product terms vanish since $\sum_{t=1}^T(y_t - \mathbf{x}_t'\mathbf{b})\mathbf{x}_t' = \mathbf{0}'$.

f.) Note from result (e) that

$$\begin{aligned} \sqrt{T}\left(T^{-1}\sum_{t=1}^T \varepsilon_t^2 - \sigma^2\right) &= \sqrt{T}\left(T^{-1}\sum_{t=1}^T e_t^2 + T^{-1}\sum_{t=1}^T(\mathbf{x}_t'\mathbf{b} - \mathbf{x}_t'\boldsymbol{\beta})^2 - \sigma^2\right) \\ &= \sqrt{T}\left(T^{-1}\sum_{t=1}^T e_t^2 - \sigma^2\right) + T^{-1/2}\sum_{t=1}^T(\mathbf{x}_t'\mathbf{b} - \mathbf{x}_t'\boldsymbol{\beta})^2. \end{aligned}$$

Applying results (a) and (d) then gives $\sqrt{T}\left(T^{-1}\sum_{t=1}^T e_t^2 - \sigma^2\right) \xrightarrow{L} N(0, 2\sigma^4)$. But

$$\sqrt{T}\left((T-k)^{-1}\sum_{t=1}^T e_t^2 - \sigma^2\right) = \left(\frac{T}{T-k}\right)\sqrt{T}\left(T^{-1}\sum_{t=1}^T e_t^2 - \sigma^2\right) + \sqrt{T}\left(\frac{k}{T-k}\right)\sigma^2$$

Since $T/(T-k) \to 1$ and $k\sqrt{T}/(T-k) \to 0$ we then also have $\sqrt{T}(s^2 - \sigma^2) \xrightarrow{L} N(0, 2\sigma^4)$.

g.) We calculate $\sqrt{T}\,(s^2 - 2)/\sqrt{8}$. If it's bigger than 2 in absolute value, we reject the null hypothesis at 5% level.