Answer key for the midterm in 2013

a.)

$$\underset{(T\times 1)}{\mathbf{y}} = \begin{bmatrix} y_1 \\ \vdots \\ y_T \end{bmatrix} \quad \underset{(T\times 3)}{\mathbf{X}} = \begin{bmatrix} 1 & x_{11} & x_{12} \\ \vdots & \vdots & \vdots \\ 1 & x_{T1} & x_{T2} \end{bmatrix} \quad \underset{(T\times 1)}{\boldsymbol{\varepsilon}} = \begin{bmatrix} \varepsilon_1 \\ \vdots \\ \varepsilon_T \end{bmatrix} \quad \underset{(3x1)}{\boldsymbol{\beta}} = \begin{bmatrix} \beta_0 \\ \beta_1 \\ \beta_2 \end{bmatrix}$$

b.)

$$\mathbf{b} = (\mathbf{X}'\mathbf{X})^{-1}\mathbf{X}'\mathbf{y}$$

$$s^2 = (T-3)^{-1}\sum_{t=1}^{T}(y_t - \mathbf{x}_t'\mathbf{b})^2$$

c.) $\underset{(1\times 3)}{\mathbf{R}} = (0,1,-1)$ $\quad \underset{(1\times 1)}{r} = 0$ $\quad \mathbf{Rb} - \mathbf{r} = b_1 - b_2$ (scalar)

Let $\hat{\mathbf{Q}} = T^{-1}\sum_{t=1}^{T}\mathbf{x}_t\mathbf{x}_t'$ $\quad \hat{q} = \mathbf{R}\hat{\mathbf{Q}}^{-1}\mathbf{R}'$

$$F = \frac{(b_1 - b_2)^2}{s^2\hat{q}/T} \sim F(1, T-3)$$

d.)

$$t = \frac{b_1 - b_2}{\sqrt{s^2\hat{q}/T}} \sim \text{ Student } t(T-3)$$

e.)

$$\underset{(T\times 2)}{\tilde{\mathbf{X}}} = \begin{bmatrix} 1 & (x_{11} + x_{12}) \\ \vdots & \vdots \\ 1 & (x_{T1} + x_{T2}) \end{bmatrix}$$

$$\begin{bmatrix} \hat{\beta}_0 \\ \hat{\beta}_1 \end{bmatrix} = (\tilde{\mathbf{X}}'\tilde{\mathbf{X}})^{-1}\tilde{\mathbf{X}}'\mathbf{y}$$

$$\hat{\beta}_2 = \hat{\beta}_1$$

f.) Notice that

$$\begin{aligned} E(\varepsilon_t\mathbf{x}_t|\mathbf{x}_t, \varepsilon_{t-1}, \mathbf{x}_{t-1}, \varepsilon_{t-2}, \mathbf{x}_{t-2}, \ldots, \varepsilon_1, \mathbf{x}_1) &= \mathbf{x}_t E(\varepsilon_t|\mathbf{x}_t, \varepsilon_{t-1}, \mathbf{x}_{t-1}, \varepsilon_{t-2}, \mathbf{x}_{t-2}, \ldots, \varepsilon_1, \mathbf{x}_1) \\ &= \mathbf{x}_t 0 \\ &= \mathbf{0}. \end{aligned}$$

So by Law of Iterated Expectations,

$$E(\varepsilon_t\mathbf{x}_t|\varepsilon_{t-1}\mathbf{x}_{t-1}, \varepsilon_{t-2}\mathbf{x}_{t-2}, \ldots, \varepsilon_1\mathbf{x}_1) = \mathbf{0}$$

Similarly $E(\varepsilon_t^2\mathbf{x}_t\mathbf{x}_t'|\mathbf{x}_t) = \sigma^2\mathbf{x}_t\mathbf{x}_t'$ so by Law of Iterated Expectations,

$$E(\varepsilon_t^2\mathbf{x}_t\mathbf{x}_t') = \sigma^2 E(\mathbf{x}_t\mathbf{x}_t') = \sigma^2\mathbf{Q}$$

g.) Write

$$F_T = \frac{(b_{1,T} - b_{2,T})^2}{s_T^2\hat{q}_T/T} = \frac{\left[\mathbf{R}\sqrt{T}(\mathbf{b}_T - \boldsymbol{\beta})\right]^2}{s_T^2\hat{q}_T}$$

Denominator:

$$s_T^2 \xrightarrow{p} \sigma^2 \text{ (as shown in class)}$$

$$\hat{q}_T = \mathbf{R}\hat{\mathbf{Q}}^{-1}\mathbf{R} \xrightarrow{p} \mathbf{R}\mathbf{Q}^{-1}\mathbf{R} = q$$

Numerator:

$$\mathbf{b} = \boldsymbol{\beta} + \left(\sum_{t=1}^{T} \mathbf{x}_t\mathbf{x}_t'\right)^{-1}\left(\sum_{t=1}^{T} \mathbf{x}_t\varepsilon_t\right)$$

$$\sqrt{T}\,(\mathbf{b}_T - \boldsymbol{\beta}) = \left(T^{-1}\sum_{t=1}^{T} \mathbf{x}_t\mathbf{x}_t'\right)^{-1}\left(T^{-1/2}\sum_{t=1}^{T} \mathbf{x}_t\varepsilon_t\right)$$

$$\xrightarrow{L} \mathbf{Q}^{-1}\mathbf{z}$$

for $\mathbf{z} \sim N(\mathbf{0}, \sigma^2\mathbf{Q})$. Hence

$$\sqrt{T}\,(\mathbf{b}_T - \boldsymbol{\beta}) \xrightarrow{L} N(\mathbf{0}, \sigma^2\mathbf{Q}^{-1})$$

$$\mathbf{R}\sqrt{T}\,(\mathbf{b}_T - \boldsymbol{\beta}) \xrightarrow{L} N(0, \sigma^2\mathbf{R}\mathbf{Q}^{-1}\mathbf{R}') = N(0, \sigma^2 q).$$

So F_T converges in distribution to square of a $N(0, \sigma^2 q)$ divided by $\sigma^2 q$, meaning

$$F_T \xrightarrow{L} \chi^2(1)$$

h.) Note $E(\varepsilon_t^2\mathbf{x}_t\mathbf{x}_t'|\mathbf{x}_t) = g(\mathbf{x}_t)\mathbf{x}_t\mathbf{x}_t'$ so by Law of Iterated Expectations,

$$E(\varepsilon_t^2\mathbf{x}_t\mathbf{x}_t') = E[g(\mathbf{x}_t)\mathbf{x}_t\mathbf{x}_t'] = \underset{(3\times 3)}{\mathbf{S}}$$

for which the natural estimate is $\hat{\mathbf{S}} = T^{-1}\sum_{t=1}^{T} e_t^2\mathbf{x}_t\mathbf{x}_t'$ for $e_t = y_t - \mathbf{x}_t'\mathbf{b}$. In this case

$$\sqrt{T}\,(\mathbf{b}_T - \boldsymbol{\beta}) \xrightarrow{L} N(\mathbf{0}, \mathbf{Q}^{-1}\mathbf{S}\mathbf{Q}^{-1})$$

$$\mathbf{R}\sqrt{T}\,(\mathbf{b}_T - \boldsymbol{\beta}) \xrightarrow{L} N(0, q^*) \quad \text{for } q^* = \mathbf{R}\mathbf{Q}^{-1}\mathbf{S}\mathbf{Q}^{-1}\mathbf{R}'$$

and we should use

$$F^* = \frac{(b_1 - b_2)^2}{\hat{q}^*/T} \quad \text{for } \hat{q}^* = \mathbf{R}\hat{\mathbf{Q}}^{-1}\hat{\mathbf{S}}\hat{\mathbf{Q}}^{-1}\mathbf{R}'$$