Answer key for the midterm in 2012

1a.) $\mathbf{x}_t = (y_{t-1}, z_t, z_{t-1})'$ $\boldsymbol{\beta} = (\phi, \alpha, \gamma)'$

$$\begin{bmatrix} \hat{\phi} \\ \hat{\alpha} \\ \hat{\gamma} \end{bmatrix} = \left(\sum_{t=1}^{T} \mathbf{x}_t \mathbf{x}_t'\right)^{-1} \left(\sum_{t=1}^{T} \mathbf{x}_t y_t\right)$$

$$\hat{\sigma}^2 = (T-3)^{-1} \sum_{t=1}^{T} (y_t - \mathbf{x}_t'\hat{\boldsymbol{\beta}})^2$$

b.) Let $\mathbf{R} = (0,1,1)'$ and use

$$\frac{\hat{\alpha} + \hat{\gamma}}{\sqrt{s^2 \mathbf{R}(\sum \mathbf{x}_t \mathbf{x}_t')^{-1} \mathbf{R}'}}.$$

Treat as Student t with $T-3$ degrees of freedom. This is only an approximation since $\mathbf{x}_t$ includes the lagged value of y.

c.) SSR_U is $T-3$ times $\hat{\sigma}^2$ above. To calculate SSR_R estimate

$$y_t = \phi y_{t-1} + \alpha(z_t - z_{t-1}) + \varepsilon_t$$

and treat

$$\frac{SSR_R - SSR_U}{SSR_U/(T-3)}$$

as $F(1, T-3)$. The F statistic in (c) is exactly equal to the square of the t statistic in (b).

2a.) $\mathbf{S} = E\{E(\varepsilon_t^2|\mathbf{x}_t)\mathbf{x}_t\mathbf{x}_t'\} = E(\sigma^2 x_{1t}\mathbf{x}_t\mathbf{x}_t') = \sigma^2\boldsymbol{\Lambda}$ where row i column j element of $\boldsymbol{\Lambda}$ is given by λ_{1ij}.

b.)

$$\begin{aligned} \mathbf{b} &= \left(\sum \mathbf{x}_t\mathbf{x}_t'\right)^{-1}\left(\sum \mathbf{x}_t y_t\right) \\ &= \boldsymbol{\beta} + \left(\sum \mathbf{x}_t\mathbf{x}_t'\right)^{-1}\left(\sum \mathbf{x}_t \varepsilon_t\right) \\ \sqrt{T}(\mathbf{b}-\boldsymbol{\beta}) &= \left(T^{-1}\sum \mathbf{x}_t\mathbf{x}_t'\right)^{-1}\left(T^{-1/2}\sum \mathbf{x}_t\varepsilon_t\right) \\ &\xrightarrow{L} \mathbf{Q}^{-1}\mathbf{z} \end{aligned}$$

where $\mathbf{z} \sim N(0, \mathbf{S})$. Thus

$$\sqrt{T}(\mathbf{b}-\boldsymbol{\beta}) \xrightarrow{L} N(\mathbf{0}, \mathbf{Q}^{-1}\mathbf{S}\mathbf{Q}^{-1}).$$

c.)

$$\begin{aligned} \hat{\boldsymbol{\beta}}_{GLS} &= \left(\sum \tilde{\mathbf{x}}_t\tilde{\mathbf{x}}_t'\right)^{-1}\left(\sum \tilde{\mathbf{x}}_t\tilde{y}_t\right) \\ \tilde{\mathbf{x}}_t &= x_{1t}^{-1/2}\tilde{\mathbf{x}}_t \\ \tilde{y}_t &= x_{1t}^{-1/2} y_t \end{aligned}$$

3a.)

$$\begin{bmatrix} \hat{\alpha} \\ \hat{\gamma} \end{bmatrix} = \left(T^{-1} \sum \begin{bmatrix} 1 \\ y_{2t} \end{bmatrix} \begin{bmatrix} 1 & y_{2t} \end{bmatrix} \right)^{-1} \left(T^{-1} \sum \begin{bmatrix} 1 \\ y_{2t} \end{bmatrix} y_{1t} \right)$$

$$= \begin{bmatrix} 1 & T^{-1} \sum y_{2t} \\ T^{-1} \sum y_{2t} & T^{-1} \sum y_{2t}^2 \end{bmatrix}^{-1} \begin{bmatrix} T^{-1} \sum y_{1t} \\ T^{-1} \sum y_{2t} y_{1t} \end{bmatrix}$$

$$\xrightarrow{p} \begin{bmatrix} 1 & E(y_{2t}) \\ E(y_{2t}) & E(y_{2t}^2) \end{bmatrix}^{-1} \begin{bmatrix} E(y_{1t}) \\ E(y_{2t} y_{1t}) \end{bmatrix}$$

$$= \begin{bmatrix} 1 & \mu_2 \\ \mu_2 & \sigma_{22} + \mu_2^2 \end{bmatrix}^{-1} \begin{bmatrix} \mu_1 \\ \sigma_{21} + \mu_2 \mu_1 \end{bmatrix}$$

$$= \begin{bmatrix} \mu_1 - \mu_2 \sigma_{21} / \sigma_{22} \\ \sigma_{21} / \sigma_{22} \end{bmatrix}$$

$$= \begin{bmatrix} \alpha_0 \\ \gamma_0 \end{bmatrix}$$

b.) α_0 and γ_0 are the values for which $E(y_{1t} - \alpha - \gamma y_{2t})^2$ are smallest, that is, they are the population parameters that would provide the best linear predictor of y_{1t} given y_{2t}.