Answer key for the midterm in 2009

1. a.)

$$\hat{\beta}_{OLS} = (\mathbf{X}'\mathbf{X})^{-1}\mathbf{X}'\mathbf{y}$$

b.)

$$\hat{\beta}_{GLS} = (\mathbf{X}'\mathbf{V}^{-1}\mathbf{X})^{-1}\mathbf{X}'\mathbf{V}^{-1}\mathbf{y}$$

c.)

$$R^2_{OLS} \geq R^2_{GLS}$$

because

$$\hat{\beta}_{OLS} = \arg\min_{\beta} \sum_{t=1}^{T} (y_t - \mathbf{x}_t'\boldsymbol{\beta})^2.$$

d.) The appropriate test statistic is

$$m^{-1}\left(\mathbf{R}\hat{\beta}_{GLS} - \mathbf{r}\right)'[s^2\mathbf{R}(\mathbf{X}'\mathbf{V}^{-1}\mathbf{X})^{-1}\mathbf{R}']^{-1}\left(\mathbf{R}\hat{\beta}_{GLS} - \mathbf{r}\right),$$

where

$$s^2 = \frac{(\mathbf{y} - \mathbf{X}\hat{\beta}_{GLS})'\mathbf{V}^{-1}(\mathbf{y} - \mathbf{X}\hat{\beta}_{GLS})}{T-k}$$

Under the null hypothesis, this statistic follows the exact $F(m, T-k)$ distribution.

2. a.)

$$\begin{aligned}
\mathbf{b}_t &= \beta + \left(\sum_{t=1}^{T} \mathbf{x_t x_t'}\right)^{-1}\left(\sum_{t=1}^{T} \mathbf{x_t \boldsymbol{\varepsilon}_t}\right) \\
&= \beta + \left(T^{-1}\sum_{t=1}^{T} \mathbf{x_t x_t'}\right)^{-1}\left(T^{-1}\sum_{t=1}^{T} \mathbf{x_t \boldsymbol{\varepsilon}_t}\right) \\
&\overset{p}{\to} \beta,
\end{aligned}$$

because $T^{-1}\sum_{t=1}^{T} \mathbf{x_t x_t'} \overset{p}{\to} \mathbf{Q}$ and $T^{-1}\sum_{t=1}^{T} \mathbf{x_t \boldsymbol{\varepsilon}_t} \overset{p}{\to} E(\varepsilon_t \mathbf{x}_t) = 0.$

b.) As

$$\sqrt{T}(\mathbf{b}_T - \boldsymbol{\beta}) = \left(T^{-1}\sum_{t=1}^{T} \mathbf{x_t x_t'}\right)^{-1}\left(T^{-1/2}\sum_{t=1}^{T} \mathbf{x_t \boldsymbol{\varepsilon}_t}\right),$$

$$\sqrt{T}(\mathbf{b}_t - \beta) \overset{L}{\to} Q^{-1}\mathbf{z}, \text{ where } \mathbf{z} \sim N(\mathbf{0}, \mathbf{S})$$

or

$$\sqrt{T}(\mathbf{b}_t - \beta) \overset{L}{\to} N(\mathbf{0}, \mathbf{Q}^{-1}\mathbf{S}\mathbf{Q}^{-1})$$

c.)

$$\hat{\mathbf{Q}} = T^{-1}\sum_{t=1}^{T} \mathbf{x_t x_t'} \text{ and } \hat{\mathbf{S}} = T^{-1}\sum_{t=1}^{T} e_t^2 \mathbf{x_t x_t'}$$

d.)

$$F_T = \frac{b_1^2}{T^{-1}\mathbf{e_1'}\hat{\mathbf{Q}}^{-1}\hat{\mathbf{S}}\hat{\mathbf{Q}}^{-1}\mathbf{e_1}}, \text{ where } \mathbf{e_1} = 1^{st} \text{ column of } I_k.$$

e.) By LLNs, CLT, Continuous Mapping Theorem and Slutsky's theorem,

$$F_T = \left(\frac{\sqrt{T}\, b_1}{\sqrt{\mathbf{e}_1' \mathbf{Q}^{-1} \mathbf{S} \mathbf{Q}^{-1} \mathbf{e}_1}} \right)^2 \frac{\mathbf{e}_1' \mathbf{Q}^{-1} \mathbf{S} \mathbf{Q}^{-1} \mathbf{e}_1}{\mathbf{e}_1' \hat{\mathbf{Q}}^{-1} \hat{\mathbf{S}} \hat{\mathbf{Q}}^{-1} \mathbf{e}_1}$$
$$\stackrel{L}{\to} \chi^2(1),$$

f.) By the delta method,

$$\sqrt{T}\,(\mathbf{g}(\mathbf{b}) - \mathbf{g}(\boldsymbol{\beta}_0)) \stackrel{L}{\to} N(\mathbf{0}, \mathbf{H}\mathbf{Q}^{-1}\mathbf{S}\mathbf{Q}^{-1}\mathbf{H}'), \text{ where } \mathbf{H} = \left.\frac{\partial \mathbf{g}}{\partial \beta'}\right|_{\beta=\beta_0}$$

Therefore, under the null ($\mathbf{g}(\boldsymbol{\beta}) = \mathbf{0}$) the statistic below is asymptotically following the $\chi^2(m)$ distribution

$$T[\mathbf{g}(\mathbf{b})]' \left[\hat{\mathbf{H}} \hat{\mathbf{Q}}^{-1} \hat{\mathbf{S}} \hat{\mathbf{Q}}^{-1} \hat{\mathbf{H}}' \right]^{-1} [\mathbf{g}(\mathbf{b})], \text{ where } \hat{\mathbf{H}} = \left.\frac{\partial \mathbf{g}}{\partial \beta'}\right|_{\beta=\mathbf{b}}$$