Econ 2142, Fall 2013

Practice Final Exam Questions

DIRECTIONS: No books, notes, or calculators of any kind are allowed.

1.) A Gaussian linear state-space model is written in the form

$$\boldsymbol{\xi}_{t+1} = \mathbf{F}\boldsymbol{\xi}_t + \mathbf{v}_{t+1}$$

$$\mathbf{y}_t = \mathbf{A}'\mathbf{x}_t + \mathbf{H}'\boldsymbol{\xi}_t + \mathbf{w}_t$$

$$\begin{bmatrix} \mathbf{v}_t \\ \mathbf{w}_t \end{bmatrix} \sim \text{i.i.d. } N\left(\begin{bmatrix} \mathbf{0} \\ \mathbf{0} \end{bmatrix}, \begin{bmatrix} \mathbf{Q} & \mathbf{0} \\ \mathbf{0} & \mathbf{R} \end{bmatrix} \right)$$

for $\mathbf{x}_t$ a vector of deterministic variables, $\mathbf{y}_t$ an observed $(n \times 1)$ vector, and $\boldsymbol{\xi}_t$ an unobserved $(r \times 1)$ vector. The Kalman filter is a recursion of the form

$$\mathbf{P}_{t+1|t} = \mathbf{F}[\mathbf{P}_{t|t-1} - \mathbf{P}_{t|t-1}\mathbf{H}(\mathbf{H}'\mathbf{P}_{t|t-1}\mathbf{H} + \mathbf{R})^{-1}\mathbf{H}'\mathbf{P}_{t|t-1}]\mathbf{F}' + \mathbf{Q}$$

$$\hat{\boldsymbol{\xi}}_{t+1|t} = \mathbf{F}\hat{\boldsymbol{\xi}}_{t|t-1} + \mathbf{F}\mathbf{P}_{t|t-1}\mathbf{H}(\mathbf{H}'\mathbf{P}_{t|t-1}\mathbf{H} + \mathbf{R})^{-1}(\mathbf{y}_t - \mathbf{A}'\mathbf{x}_t - \mathbf{H}'\hat{\boldsymbol{\xi}}_{t|t-1}).$$

a.) Calculate the forecast $\hat{\mathbf{y}}_{t|t-1}$ and its mean squared error $\mathbf{C}_{t|t-1} = E(\mathbf{y}_t - \hat{\mathbf{y}}_{t|t-1})(\mathbf{y}_t - \hat{\mathbf{y}}_{t|t-1})'$ as functions of $\mathbf{P}_{t|t-1}$ and $\hat{\boldsymbol{\xi}}_{t|t-1}$.

b.) Use your results from part (a) to write an expression for the log likelihood function

$$\mathcal{L}(\boldsymbol{\theta}) = \textstyle\sum_{t=1}^{T} \log f(\mathbf{y}_t|\Omega_{t-1};\boldsymbol{\theta})$$

for $\Omega_t = \{\mathbf{y}_t, \mathbf{y}_{t-1}, ..., \mathbf{y}_1\}$ the set of observations through date t and $\boldsymbol{\theta}$ a vector containing the unknown elements of $\mathbf{F}, \mathbf{Q}, \mathbf{A}, \mathbf{H}, \mathbf{R}$.

c.) A Gaussian MA(2) process for a scalar y_t is characterized by

$$y_t = \mu + \varepsilon_t + \theta_1\varepsilon_{t-1} + \theta_2\varepsilon_{t-2}$$

for $\varepsilon_t \sim$ i.i.d. $N(0, \sigma^2)$. Show that this can be written in state-space form by specifying $\boldsymbol{\xi}_t = (\varepsilon_t, \varepsilon_{t-1}, \varepsilon_{t-2})'$ and $\mathbf{x}_t = 1$. Find the values of $\mathbf{F}, \mathbf{Q}, \mathbf{A}, \mathbf{H}, \mathbf{R}$ for this representation.

d.) For the MA(2) example in part (c), what values would you use for $\hat{\boldsymbol{\xi}}_{1|0}$ and $\mathbf{P}_{1|0}$ to start the iteration?

2.) Consider a first-order structural vector autoregression of the form

$$\mathbf{B}_0\mathbf{y}_t = \mathbf{b} + \mathbf{B}_1\mathbf{y}_{t-1} + \mathbf{u}_t$$

where $\mathbf{u}_t \sim$ i.i.d. $N(\mathbf{0}, \mathbf{D})$ is a vector of uncorrelated structural shocks. Here $\mathbf{y}_t$ is a vector whose first element consists of the gap between actual and potential real GDP, the second element is the annual inflation rate, and the third element is the fed funds rate. The third element of $\mathbf{u}_t$ is viewed as a shock to monetary policy, and the object of interest is presumed to be estimating the vector

$$\frac{\partial \mathbf{y}_{t+s}}{\partial u_{3t}}.$$

a.) Suppose we were using monthly data and believed that a change in monetary policy (that is, a change in u_{3t}) takes longer than one month to have any effects on output or inflation. Describe a strategy you could use to estimate $\partial \mathbf{y}_{t+s}/\partial u_{3t}$.

b.) Describe an alternative approach to identification that does not assume a one-month delay but instead relies on higher-frequency data, even though you do not have observations on output and inflation at any frequencies higher than one month.

3.) Consider the following regression

$$\Delta y_t = c + \eta y_{t-1} + \zeta \Delta y_{t-1} + \varepsilon_t.$$

You are asked to consider the properties of an OLS regression under the null hypothesis that the true values of the coefficients satisfy $c = \eta = 0$ and $\zeta = \zeta_0$ where $0 < \zeta_0 < 1$.

a.) State (but do not derive) the plims of $\hat{c}$, $\hat{\eta}$, and $\hat{\zeta}$ under the above assumptions.

b.) State in general terms (you do not need to use any equations or derivations) the asymptotic distributions of the OLS t statistics $\hat{\eta}/\hat{\sigma}_{\hat{\eta}}$ and $(\hat{\zeta} - \zeta_0)/\hat{\sigma}_{\hat{\zeta}}$.

c.) What is the relation between the OLS estimates $\hat{c}, \hat{\eta}, \hat{\zeta}$ and OLS estimates of k, ϕ_1, and ϕ_2 in the OLS regression

$$y_t = k + \phi_1 y_{t-1} + \phi_2 y_{t-2} + \varepsilon_t?$$

d.) State in general terms (you do not need to use any equations or derivations) the asymptotic distributions of the OLS t statistic $(\hat{\phi}_1 - \phi_1^{(0)})/\hat{\sigma}_{\hat{\phi}_1}$ where $\phi_1^{(0)}$ denotes the true value.

4.) Consider the following model. The behavior of an observed scalar y_t depends on the value of an unobserved integer s_t where s_t takes on one of the values $\{1, 2, ..., N\}$:

$$y_t = \begin{cases} \mu_1 + \varepsilon_t & \text{when } s_t = 1 \\ \mu_2 + \varepsilon_t & \text{when } s_t = 2 \\ \vdots & \vdots \\ \mu_N + \varepsilon_t & \text{when } s_t = N \end{cases}.$$

Here $\varepsilon_t \sim$ i.i.d. $N(0, \sigma^2)$ and s_t follows a Markov chain that is independent of $\{\varepsilon_t\}$:

$$\text{Prob}(s_t = j | s_{t-1} = i) = p_{ij}.$$

a.) Define the $(N \times 1)$ vector to take on the value $\boldsymbol{\xi}_t = \mathbf{e}_j$ (column j of $\mathbf{I}_N$) when $s_t = j$. For example, $\boldsymbol{\xi}_t = (1, 0, ..., 0)'$ when $s_t = 1$. Write the equation of motion for $\boldsymbol{\xi}_t$ in the form of a first-order vector autoregression.

b.) Suppose you have calculated an inference $\hat{\boldsymbol{\xi}}_{t|t} = E(\boldsymbol{\xi}_t | \Omega_t)$ for $\Omega_t = \{y_t, y_{t-1}, ..., y_1\}$. Explain in words the meaning of the jth element of $\hat{\boldsymbol{\xi}}_{t|t}$.

c.) Use your results from (a) and (b) to calculate $\hat{\boldsymbol{\xi}}_{t+1|t} = E(\boldsymbol{\xi}_{t+1} | \Omega_t)$.

d.) Let $\boldsymbol{\eta}_{t+1}$ denote an $(N \times 1)$ vector whose jth element is $p(y_{t+1} | \boldsymbol{\xi}_{t+1})$. Calculate the value of $\boldsymbol{\eta}_{t+1}$.

e.) Use your results from (c) and (d) to calculate an $(N \times 1)$ vector whose jth element is the joint probability of y_{t+1} and $s_{t+1} = j$ conditional on Ω_t.

f.) Use your result from (e) to calculate the likelihood $p(y_{t+1} | \Omega_t)$.

5.) Consider the following nonlinear state-space model,

$$\xi_{t+1} = \phi \xi_t + v_{t+1} \tag{1}$$

$$y_t = \exp(\alpha \xi_t) + w_t \tag{2}$$

where the scalars v_t and w_t are characterized by

$$\begin{bmatrix} v_t \\ w_t \end{bmatrix} \sim \text{i.i.d. } N\left(\begin{bmatrix} 0 \\ 0 \end{bmatrix}, \begin{bmatrix} Q & 0 \\ 0 & R \end{bmatrix} \right).$$

a.) Assume you know the values of ϕ, α, Q, R and have calculated a forecast of ξ_t based on observation of $\Omega_{t-1} = \{y_{t-1}, y_{t-2}, ..., y_1\}$, denoted $\hat{\xi}_{t|t-1}$ along with its mean squared error, $P_{t|t-1} = E(\xi_t - \hat{\xi}_{t|t-1})^2$. Use a first-order Taylor approximation of (2) (known as the extended Kalman filter) to calculate a forecast $\hat{y}_{t|t-1}$ and its mean squared error.

b.) Given $\hat{\xi}_{t|t-1}$ and your answer to (a), show how observation of y_t can be used to calculate an updated forecast $\hat{\xi}_{t+1|t}$ and its MSE $P_{t+1|t}$. Hint: you might find some of the formulas given at the start of question (1) helpful.

6.) This question considers the same nonlinear model in question (5) but now explores inference using a bootstrap particle filter a.) Assume you know the values of ϕ, α, Q, R and have generated a set of particles $\Lambda_t^{(i)} = \{\xi_t^{(i)}, \xi_{t-1}^{(i)}, ..., \xi_1^{(i)}\}$ for $i = 1, ..., D$ with associated weights $\omega_t^{(i)}$ where $\sum_{i=1}^{D} \omega_t^{(i)} = 1$. Recall that the weights are proportional to

$$\omega_t^{(i)} \propto \frac{p_t(\Lambda_t^{(i)}|\Omega_t)}{g_t(\Lambda_t^{(i)}|\Omega_{t-1})}$$

where $p_t(\Lambda_t^{(i)}|\Omega_{t-1})$ is the probability that Λ_t took on the value $\Lambda_t^{(i)}$ conditional on Ω_t and $g_t(\Lambda_t^{(i)}|\Omega_{t-1})$ is the importance density you used to generate the set of particles $\{\Lambda_t^{(i)}\}_{i=1}^{D}$ evaluated at the point $\Lambda_t^{(i)}$. Describe a simple way you could use the state equation (1) to generate a candidate value for $\xi_{t+1}^{(i)}$ and give the expression for $g_{t+1}(\Lambda_{t+1}^{(i)}|\Omega_t)$.

b.) Calculate an expression for $\omega_{t+1}^{(i)}$ where

$$\omega_{t+1}^{(i)} \propto \frac{p(y_{t+1}|\xi_{t+1}^{(i)})p(\xi_{t+1}^{(i)}|\xi_t^{(i)})p_t(\Lambda_t^{(i)}|\Omega_t)}{g_{t+1}(\Lambda_{t+1}^{(i)}|\Omega_t)}.$$

Your answer should express $\omega_{t+1}^{(i)}$ as a function of $\Lambda_{t+1}^{(i)}, \omega_t^{(i)}, y_{t+1}, \phi, \alpha, Q, R$.

c.) Explain how you could use your result from part (b) to calculate an estimate of $E(\xi_{t+1}|\Omega_{t+1})$.