Unexpected Utility: Testing The Assumptions on Risk Preferences Using Convex Budgets

Preliminary Notes: Not for Circulation

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Abstract

We conduct laboratory experiments on risky choices using convex budgets. The only assumption is that utility depends on the probability $p$ of a (gain or loss) $x$, $u = u(p, x)$. Individuals then choose $p$ and $x$ on a linear budget constraint of the form $r_1 p + r_2 x = m$. Our methodology allows us to test the most important assumptions applied to risk preferences: rationality of choice, the independence axiom, prospect theory, probability weighting, and constant relative risk aversion. We find, on average, an unexpectedly strong coherence with standard risk aversive expected utility and CRRA utility functions over gains, prospect-theory risk-loving over losses, but not probability weighting. Moreover, choices are largely consistent with a rational choice model of risky choices, especially over gains.
1. Introduction

We study behavior under risk using a very general model which treats potential gains or losses and their probabilities directly, as arguments in a utility function. We apply revealed preference theory to develop restrictions about rational choices in this framework. We report tests of those restrictions, as well as those from more restrictive models of risky choice, such as Expected Utility theory, using experimental data.

The usual economic approach to decision making with risk is to apply expectations operators to a utility function to get expected utility. Restrictions on allowable choices can be derived even with general assumptions about the shape of the utility function. Empirical work in this field involves testing if those restrictions are met, and then either estimating a utility function, or weakening the assumptions, for example by allowing for the use of nonlinear probability weights, or different utility functions for gains and losses. This work generally finds that EU does not explain important aspects of choice behavior. (See Allais, K&T, Rabin)

The approach in this paper is different. We start with very weak assumptions: a potential gain is a good, as is the probability of a gain. Potential losses and their probabilities are bads. We do not apply expectations operators, instead we define utility functions directly over these goods and bads. We then explore the restrictions that revealed preference theory imposes on choices, given these assumptions. In particular, we show that in this framework the Generalized Axiom of Revealed Preference (GARP) can be used to rule out certain combinations of risky choices. We develop an experimental procedure for collecting the data needed to test for these sorts of GARP violations, and we show that most, though not all people, make decisions over risky choices that are consistent with GARP.

We then use this setup to investigate the restrictions imposed by other models of risky decision making and show how they can be tested. In particular, we show what EV maximization, risk aversion, CRRA utility maximization, SEU, and prospect theory (cumulative?) imply about decisions in our setup. We use our data to report on the extent to which the results of our experiments are consistent with these models of behavior.
2. Model

Consider a gamble with two possible outcomes, $x$ and $y$. Let $p$ be the probability of winning a good prize, $x$. Let $y$ be the bad outcome, worth something less than $x$.

Then the expected value of the gamble is

$$EV = px$$

Suppose utility is $u(x)$. Normalize the utility of the bad outcome to be 0. Then expected utility is

$$EU = pu(x)$$

For illustration, let $u(x) = x^a$, where $0 < a < 1$. This is the Constant Relative Risk Aversion (CRRA) utility function. Expected utility is then

$$EU = px^a.$$ 

Suppose we gave an individual a “budget” of gambles $(p, x)$ that are on a line and ask them to optimize. That is

$$\max_{p,x} px^a$$

s.t. \( r_1p + r_2x = m \)

where $r_1$ and $r_2$ act as prices of $p$ and $x$, respectively. Notice that is simply as maximizing a Cobb-Douglas utility function with solutions

$$p^*(c, m) = \frac{1}{1 + a r_1} \frac{m}{r_1}$$

$$x^*(c, m) = \frac{a}{1 + a r_2} \frac{m}{r_2}.$$

A plausible estimate from experiments is $a = 1/2$. If that’s true, then we can expect

$$p^* = \frac{2}{3} \frac{m}{r_1}$$

$$x^* = \frac{1}{3} \frac{m}{r_2}.$$ 

So, for instance, if $r_1 = r_2 = m = 1$ a person would choose $p = 2/3$ and $x = 1/3$. The gamble that would maximize $EV$, however, is $p = x = 1/2$. Hence, risk
aversion is making the person choose a gamble with a higher likelihood of winning a lower prize. Stated differently, for any point in \((p, x)\)-space, the iso-EU curve through that point will always be flatter than the iso-EV curve, but both curves will be convex under the assumption of diminishing marginal utility. Below is a picture of the situation.

![Diagram](image_url)

Figure 1: Choosing gambles along a linear budget.

2.1. Risk Aversion and Risk Loving over Gains

Consider a risk neutral person who simply maximizes expected value. Then this person will choose

\[
\max_{p,x} px \\
\text{s.t. } r_1 p + r_2 x = m
\]
Substituting in the budget yields

$$\max_p \left(p \left( \frac{m - r_1 p}{r_2} \right) \right).$$

This then yields the EV maximizing $p$, say $\hat{p}$:

$$\hat{p} = \frac{m}{2} \frac{1}{r_1}.$$

Now consider an expected utility maximizer.

$$\max_p \left( p u \left( \frac{m - r_1 p}{r_2} \right) \right).$$

The first order conditions is

$$u() - pu'() \frac{r_1}{r_2} = 0. \tag{2.1}$$

Suppose the individual is risk averse, that is, $u()$ is concave. Then we know that $u(x) - xu'(x) > u(0)$. By assumption, $u(0) = 0$. Substituting this into (2.1) we get that

$$0 > \frac{m - r_1 p}{r_2} u'() - pu'() \frac{r_1}{r_2}$$

$$p^*_{ra} > \frac{m}{2} \frac{1}{r_1} = \hat{p},$$

that is, the EU maximizing $p$, $p^*_{ra}$, is above the expected value maximizing $p$. The intuitive reasoning is that at the expected value maximizing outcome, both $p$ and $x$ are contributing equally to the value, but both are not equally weighted in utility. At the margin, $p$ is more valuable.

Suppose instead that the person is risk loving, so that $u()$ is convex. Then we know that $u(x) - xu'(x) < u(0)$. Again substitution into (2.1), we get a similar result but reversed:

$$p^*_{rl} < \frac{m}{2} \frac{1}{r_1} = \hat{p}.$$  

This time the intuition is that, at the margin, $x$ is valued more than $p$.  

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2.2. Risk Aversion and Risk Loving over Losses

For losses the model works exactly the same, but just inverted. Instead of talking about expected value maximization, we talk about expected value minimization. Rather than maximizing utility, we talk about minimizing utility. Again fix the outcome \( y \) to be \( y = 0 \). Note that now \( y \) is the good outcome and the loss \( x \) is the bad outcome. Now consider a gamble with expected value \(-px\) on a line such that \( r_1p + r_2x = m \). Then the expect value maximum is simply \( p = m \) and \( x = 0 \), but the expected value minimum is \( \hat{p} = m/2r_1 \).

Consider an expected utility maximizer who is facing prospective losses on a line \( p + cx = m \). We now ask which gamble will minimize expected utility, that is, which gamble is the least preferred:

\[
\min_{p,x} pu(-x) \quad \text{s.t.} \quad r_1p + r_2x = m, \quad x \geq 0.
\]

Again substituting the budget we get

\[
\min_p pu\left(-\frac{m - r_1p}{r_2}\right).
\]

The first order condition is

\[
u() + pu'(\left\frac{r_1}{r_2}\right) = 0. \quad (2.2)
\]

Note that this can be met since \( u(-x) < u(0) = 0 \) while \( u'(x) > 0 \).

Suppose \( u() \) is concave, that is, the individual is risk averse. Then we know that \( u(x) - xu'(x) > u(0) = 0 \). Substituting this into (2.2) we get that

\[
0 = u() + pu'(\left\frac{r_1}{r_2}\right) > xu'() + pu'(\left\frac{r_1}{r_2}\right)
= -\frac{m - r_1p}{r_2} u'() + pu'(\left\frac{r_1}{r_2}\right)
\]

It follows that

\[
p_{ra}^{**} < \frac{m}{2r_1} = \hat{p},
\]

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that is, the EU minimizing $p$, $p^{**}_{ra}$, is below the expected value minimizing $p$.

Suppose instead that $u()$ is convex, so that the person is risk loving. Then we know that $u(x) - xu'(x) < u(0)$. Again substitution into (2.1), we get a similar result but reversed:

$$p^{**}_{rl} > \frac{m}{2r_1} = \hat{p}.$$  

This is illustrated in the picture below for $r_1 = 1$.

![Figure 2: Choices that reveal risk aversion and risk loving behavior for both gains and losses](image)

What this means is that we can give subjects a choice over gains, asking them to pick the gamble they most prefer, and offer them the same choices over losses and ask them which gamble they least prefer. In each case we can check whether $p^*$ and $p^{**}$ meet requirements of risk aversion and risk loving, and how many of our subjects fit prospect theory.

NOTE: This gives an extra consistency check. We can ask whether $p^* > \hat{p}$ for all budgets or $p^* < \hat{p}$ for all budgets. If not, it would indicate risk aversion
for some and risk loving for other gambles. Note, there is nothing wrong with a theory that allows each. Recall, standard theory predicts risk neutrality over small gambles. If we add some “joy of winning” then this could appear risk loving for small gambles. Could we in fact use this diagnostically? As $x$ increases, for instance, is there a switch from risk loving to risk averse?

2.3. Preferences to Estimate to test Prospect Theory

Below is a parametric model that has some desirable properties:

Let $-\infty < x < \infty$ be the varying outcome, and $y = 0$ be the fixed outcome. Then one could estimate this utility function:

$$u(x) = \begin{cases} 
    x^\alpha & \text{if } 1 < x \\
    x & \text{if } -1 \leq x \leq 1 \\
    -1 - (-x - 1)^{1+\beta} & \text{if } x < -1
\end{cases}$$

The restrictions for risk aversion are that $0 < \alpha < 1$, and $\beta > 0$. Risk loving follows from $\alpha > 1$ and $0 \leq \beta < 1$. We could estimate $\alpha$ and $\beta$ for each subject. Notice the linearization around $x = 0$. This is because $x^\alpha$ has a slope of infinity as $x$ approaches 0. We want the person to be risk neutral in the vicinity of $x = 0$.

2.4. Offer Curves: The Independence Axiom

The independence axiom is what allows us to write the optimization problem this way.

$$\max_{p,x} pu(x)$$

s.t. $r_1 p + r_2 x = m$

Consider the first order conditions as solved above:

$$p \frac{u'(x)}{u(x)} = \frac{r_2}{r_1}$$

Rearrange as .

$$pr_1 \frac{u'(x)}{u(x)} = r_2$$

Substitute in the budget constraint:

$$(m - r_2x) \frac{u'(x)}{u(x)} = r_2 \quad (2.3)$$
Notice that (2.3) can be solved for $x$ as a function of $m$ and $r_2$ alone. That is, the choice of $x$ is independent of the price or $p$. This result comes directly from the assumption that utility can be written as linear in $p$, which is the isomorphic to the independence axiom. Thus, we have a direct test of the independence axiom by looking at offer curves. Are they vertical lines?

2.5. Recovering Probability Weighting

An alternative to expected utility, which removes the independence axiom, is called probability weighting (Prelec and Tversky ???). Rather than using the objective probability, $p$, it is hypothesized that subjects act as if the probabilities are $w(p)$, and they have an objective function $U(p, x) = \sum w(p_i)u(x_i)$. The function $w(p)$ is assumed to be continuous and differentiable, with $w(0) = 1, w(1) = 1$, but for $0 \leq p \leq 1$ subjects tend to overweight low probabilities and under weight high probabilities.

Consider the simple version this problem above:

$$\max_{p,x} w(p)u(x)$$

subject to $r_1p + r_2x = m$

This yields first order conditions

$$\frac{w(p)}{w'(p)} \frac{u'(x)}{u(x)} = \frac{r_2}{r_1}.$$  \hspace{1cm} (2.4)

Compare these to the first order conditions under standard expected utility:

$$\frac{p}{u'(x)} = \frac{r_2}{r_1},$$ \hspace{1cm} (2.5)

How can we use our analysis to identify a function $w(p)$?

Note for $p$ close to 0 (but not too close), $w(p) > p$ and $w'(p) < 0$. This means $w(p)/w'(p) > p$. Let $x^w$ and $x^s$ be the solutions to the probability weighting (2.4) standard (2.5) first order conditions. Then

$$\frac{w(p)}{w'(p)} \frac{u'(x^w)}{u(x^w)} = \frac{p}{u'(x^s)} \frac{u'(x^s)}{u(x^s)}$$

but for $p$ small

$$\frac{u'(x^w)}{u(x^w)} > \frac{u'(x^s)}{u(x^s)}.$$
By risk aversion, $u'/u$ is a decreasing function, so for small $p$, $x^w < x^s$. A similar argument holds for $p$ close to 1, so $x^w > x^s$.

This gives us a test of the independence axiom relative to probability weighting. As shown in Figure 3, offer curves should be vertical if the independence axiom holds, but slope up if probability weighting holds.

![Offer Curves with and without Probability Weighting](image)

Figure 3: Offer Curves with and without Probability Weighting

2.6. CRRA Preferences

If we assume $u(x) = x^\alpha$, then expected utility is $U = px^\alpha$ which is a Cobb-Douglas utility function. So we can test this by checking for homotheticity, that is, rays from the origin through choices on parallel budgets.

Data to be presented at the lecture.