A SIMPLE CHARACTERIZATION OF RESPONSIVE CHOICE†

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Abstract. We provide several characterizations of q-responsive choice functions, based on classical axioms of matching theory and revealed preference theory.

1. Introduction

In this note, we characterize choice functions which arise in matching theory and which were studied in detail in Chambers and Yenmez (2017). Our particular interest is in choice functions for which there is a rational (strict) preference relation which always selects the q-best elements whenever available. Such choice functions are termed q-responsive. In the context of schools with preferences over students, for example, such a choice function would be one for which the school admits the q highest ranked students.

Our particular interest in this note is to understand the behavioral underpinnings of this choice theoretic model when the preference itself is not directly observable. To this end, we provide several such characterizations. Notably, we provide characterizations for choice functions which operate on the power set of a set of alternatives, but also on smaller sets of budgets.

On the power set, we can imagine a choice function which always selects q-alternatives whenever there are q available. This concept is termed q-acceptance. It turns out that, together with q-acceptance, it is enough to impose an axiom of Ehlers and Sprumont (2008), the weakened weak axiom of revealed preference (WWARP). This axiom states that for any pair of partners, x, y, if x is chosen when y is available and y is not chosen, then it can never be that y is chosen when x is available and x is not chosen. We show that a choice function is responsive for value q if and only if it is q-acceptant and satisfies WWARP (Theorem 1). This result is a direct generalization of the classical result that a single-valued choice function satisfying the weak axiom of revealed preference is classically rationalizable.

We also show how to utilize a weakened version of WWARP, applying only to sets of cardinality $q+1$, together with a familiar axiom from matching theory (substitutability) to characterize the same family.

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A different weakening of WWARP was discussed in Aleskerov et al. (2007) or Tyson (2008), and used to axiomatize a form of satisficing behavior.[1] For lack of better terminology, we call it the weakened strong axiom of revealed preference. The axiom roughly rules out strict preference cycles of any length. Aleskerov et al. (2007) and Tyson (2008) use the axiom to characterize choice functions for which there is a rational preference, and for any set of alternatives, the choice function always selects some "at least as good as set" from the set. Hence, if it selects an alternative, it must select any present alternative which is at least as good as the one selected. It is a short step to add q-acceptance to fully characterize q-responsive choice functions here.

A precedent to this note is the work of Eliaz et al. (2011), who characterize the same rules by means of different axioms. Their work provides a characterization only for choice functions defined on the power set. Our axiomatization, based on axioms which are standard in the literature, allows us to establish a characterization on choice functions which are not defined on the power set.

2. The model

Suppose $\mathcal{X}$ is a set of alternatives and $\mathcal{P}(\mathcal{X}) = 2^{\mathcal{X}}$ is the powerset of $\mathcal{X}$. Let $\Sigma \subseteq \mathcal{P}(\mathcal{X})$ be a set of ***budgets***. A ***choice function*** is a mapping $C : \Sigma \to \mathcal{P}(\mathcal{X})$ such that

- for every $X \in \Sigma$, $C(X) \subseteq X$ and
- for every $\emptyset \neq X \subseteq \mathcal{X}$, $C(X) \neq \emptyset$.

We require the chosen set to be non-empty if there is at least one alternative available.

A ***preference relation*** $\succeq$ on $\mathcal{X}$ is a binary relation on $\mathcal{X}$ that is complete, transitive, and antisymmetric.[2] Let $q \in \{1, 2, \ldots\}$ be an integer.

We first introduce the following class of choice functions.

Definition 1. *Choice function C is* ***q-responsive*** *if there is a preference relation $\succeq$ such that for all $X \in \mathcal{X}$ for which $|X| \leq q$, $C(X) = X$, and for all $X \in \mathcal{X}$ for which $|X| > q$, then $C(X)$ is defined as $C(X) = \{x_1^*, \ldots, x_q^*\}$, where $x_1^* = \arg\max_{\succeq} X$, and for all $i = 2, \ldots, k$, $x_i^* = \arg\max_{\succeq} X \setminus \{x_1^*, \ldots, x_{i-1}^*\}$.*

In other words, C is q-responsive if there is a preference relation for which C selects the highest q alternatives, whenever available. Such choice rules are used in matching literature including the seminal work of Gale and Shapley (1962) and also in practical school choice (Abdulkadiroğlu and Sönmez, 2003).

[1] In our context, q-responsive rules are a form of satisficing behavior. See Chambers and Echenique (2016) for details.

[2] Complete: For all $x, y \in \mathcal{X}$, $x \succeq y$ or $y \succeq x$. Antisymmetric: For all $x, y \in \mathcal{X}$, $x \succeq y$ and $y \succeq x$ implies $x = y$. Transitive: For all $x, y, z \in \mathcal{X}$, $x \succeq y$ and $y \succeq z$ implies $x \succeq z$.

Definition 2. *Choice function C is* ***substitutable*** *if for every $x \in X \subseteq Y$, $x \in C(Y)$ implies $x \in C(X)$.*

Substitutability requires that if an alternative is chosen from a set, then it must also be chosen from a subset containing it. Kelso and Crawford (1982); Roth (1984) were the first to study substitutability in a matching context. However, it was first studied in the choice literature and known as Sen's α or Chernoff's axiom (Moulin, 1985).

Definition 3. *Choice function C is* **q-*acceptant*** *if $|C(S)| = \min\{q, |S|\}$. Choice function C is* ***acceptant*** *if it is q-acceptant for some q.*

Acceptance means that there is a capacity such that whenever the number of alternatives is less than the capacity all the alternatives are chosen and otherwise the capacity is filled.

Definition 4. *Choice rule C on Σ satisfies the* ***weakened weak axiom of revealed preference (WWARP)*** *if, for every $x, y \in \mathcal{X}$, $X \in \Sigma$, and $Y \in \Sigma$ such that $x, y \in X \cap Y$,*

$$x \in C(X) \text{ and } y \in C(Y) \setminus C(X) \text{ imply } x \in C(Y).$$

WWARP was first introduced by Ehlers and Sprumont (2008) in the context of nonrationalizable choice, though it was implicitly discussed in Wilson (1970), who analyzed the class of choice functions satisfying it (calling them "Q cuts").

A weaker version of WWARP is also useful.

Definition 5. *Choice function C satisfies the* ***q-weakened weak axiom of revealed preference (WWARP)*** *if, for any x, y, X, and Y such that $x, y \in X \cap Y$ and $|X| = |Y| = q$,*

$$x \in C(X) \text{ and } y \in C(Y) \setminus C(X) \text{ imply } x \in C(Y).$$

The following axiom was introduced in Aleskerov et al. (2007) or Tyson (2008), and used to axiomatize a form of satisficing behavior. For lack of better terminology, we call it the weakened strong axiom of revealed preference.

Definition 6. *Choice rule C on Σ satisfies the* ***weakened strong axiom of revealed preference (WSARP)*** *if for every integer k, every $x_1, x_2, \ldots, x_k \in \mathcal{X}$, and every $X_1, \ldots, X_k \in \Sigma$ for which for all $i = 1, \ldots, k-1$, if $\{x_i, x_{i+1}\} \subseteq X_i$, $x_i \in C(X_i)$, and $x_{i+1} \notin C(X_i)$, then if $\{x_k, x_1\} \subseteq X_k$ and $x_k \in C(X_k)$, it follows that $x_1 \in C(X_k)$.*

Say that alternative x is *revealed preferred* to alternative y, when there exists a set containing both such that alternative x is chosen while alternative y is rejected. WSARP rules out any cycles in the revealed preference relation. WSARP weakens congruence of Richter (1966) or the M-structure axiom of Hansson (1968) in the same way that WWARP weakens the weak axiom of revealed preference.

3. Results

Our first result provides a characterization of q-responsive choice functions.

Theorem 1. *Suppose that $\Sigma = \mathcal{P}(\mathcal{X})$. Choice function C is q-responsive if and only if it is q-acceptant and satisfies WWARP.*

Proof. One direction is obvious. Conversely, suppose that C satisfies WWARP and it is q-acceptant. Define $\succ^*$ by $x \succ^* y$ if there exists X for which $\{x, y\} \subseteq X$, $x \in C(X)$ and $y \notin C(X)$. WWARP is equivalent to asymmetry of the relation $\succ^*$. We claim that $\succ^*$ is transitive.

Suppose that $x \succ^* y \succ^* z$. Since $x \succ^* y$, there exists X for which $\{x, y\} \subseteq X$, $x \in C(X)$ and $y \notin C(X)$. Hence, by q-acceptance, $|C(X)| = q$. Consequently, there exists $\{a_1, \ldots, a_{k-1}\} \subseteq X$ for which $C(X) = \{x, a_1, \ldots, a_{k-1}\}$. Therefore, for all $i \in \{1, \ldots, k-1\}$, $a_i \succ^* y$. Obviously, $a_i \neq y$ for all i. Now, consider $Y = \{x, y, z, a_1, \ldots, a_{k-1}\}$. Suppose, by means of contradiction, that $z \in C(Y)$. Then $y \in C(Y)$; otherwise, we have $y \succ^* z$ and $z \succ^* y$ contradicting asymmetry. Now, since $y \in C(Y)$, we have $x \in C(Y)$ and $a_i \in C(Y)$ for all i; otherwise, we would have either $x \succ^* y$ and $y \succ^* x$ or $a_i \succ^* y$ and $y \succ^* a_i$ for some i. But $|Y| \geq k+1$, so that $|C(Y)| \geq k+1$, contradicting the assumption that C is acceptant. Similarly we show that $y \notin C(Y)$. Since C is q-acceptant, this implies that $C(Y) = \{x, a_1, \ldots, a_{k-1}\}$. So $x \succ^* z$.

The rest is now standard; by the Szpilrajn Theorem (see for example Duggan (1999)), there is a preference relation $\succeq$ for which $x \succ^* y$ implies $x \succ y$. Clearly, if $x \in C(X)$ and $y \succeq x$ and $y \in X$, we have $y \in C(X)$. Otherwise, we would have $y \succ^* x$, and $y \succ x$, a contradiction. By definition C is q-responsive with respect to $\succeq$. This kind of construction was first introduced in Aleskerov et al. (2007) or Tyson (2008). $\square$

An immediate corollary is the following.

Corollary 1. *Suppose that $\Sigma = \mathcal{P}(\mathcal{X})$. Choice function C is responsive if and only if it is acceptant and satisfies WWARP.*

Responsivity in the case of $|C(X)| = 1$ is equivalent to the standard notion of rationalizability by a preference relation. And moreover, when $|C(X)| = 1$ for all X, WWARP is equivalent to the classical weak axiom of revealed preference (see for example Uzawa (1956) or Arrow (1959)). Thus, Theorem 1 is a generalization of the well-known result that the weak axiom of revealed preference characterizes rationalizability when choice is single-valued and all budgets are available.

Lemma 1. *For $\Sigma = \mathcal{P}(\mathcal{X})$, if choice function C satisfies substitutability, $q+1$-WWARP, and q-acceptance, then it satisfies WWARP.*

Proof. Suppose by means of contradiction that there are X and Y, x, y for which $x, y \in X \cap Y$, $x \in C(X)$, $y \notin C(X)$, $y \in C(Y)$, $x \notin C(Y)$. By q-acceptance, it follows that $|X| \geq q+1$ and $|Y| \geq q+1$, and by $q+1$-WWARP, at least one of these inequalities is strict.

Now, $C(C(X) \cup \{y\}) = C(X)$ by substitutability and q-acceptance; likewise, $C(C(Y) \cup \{x\}) = C(Y)$. But note that $|C(X) \cup \{y\}| = |C(Y) \cup \{x\}| = q+1$, and that $x \in C(C(X) \cup \{y\})$, $y \notin C(C(X) \cup \{y\})$, $y \in C(C(Y) \cup \{x\})$, and $x \notin C(C(Y) \cup \{x\})$, a contradiction to $q+1$-WWARP. □

Our next result provides a characterization of q-responsive choice functions using substitutability.

Theorem 2. *If $\Sigma = \mathcal{P}(\mathcal{X})$, then choice function C is q-responsive if and only if it is substitutable, satisfies $q+1$-WWARP, and q-acceptance.*

Proof. Follows immediately from Lemma 1 and Theorem 1. □

Finally, we may ask what happens when a choice function does not operate on all of $\mathcal{X}$. To this end, let $\Sigma \subseteq \mathcal{X}$.

An earlier result, due to Eliaz et al. (2011), provides another characterization of q-responsive rules. While obviously the two characterizations are mathematically equivalent, there are a few differences. First, our characterization is based on the classical revealed preference relation, while the result of Eliaz et al. (2011) does not make direct use of the concept. Our result can also be extended to a case in which the choice function is not defined on all of $\mathcal{P}(\mathcal{X})$, but rather on a subset. It is this result to which we now turn.

Theorem 3. *For any Σ, choice function C is q-responsive if and only if it is q-acceptant and satisfies WSARP.*

Proof. Follow the construction of the proof of Theorem 1. The content of WSARP is that the relation $\succ^*$ exhibits no cycles. Because of this, if $\succeq^T$ denotes the transitive closure of $\succ^*$, it follows that $\succ^* \subseteq \succ^T$. By the Szpilrajn Theorem (see for example Duggan (1999)), there is a preference relation $\succeq$ for which $x \succ^T y$ implies $x \succ y$, so that $x \succ^* y$ implies $x \succ y$. Clearly, if $x \in C(X)$ and $y \succeq x$ and $y \in X$, we have $y \in C(X)$. Otherwise, we would have $y \succ^* x$, and $y \succ x$, a contradiction. By definition C is q-responsive with respect to $\succeq$. □

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