1 Productivity Shocks with Initially Unbalanced Current Accounts

There are two periods and two countries. Home produces with \( Y = AF(K) \) and Foreign with \( Y^* = A^*F(K^*) \), where \( A \) and \( A^* \) are productivity parameters. Both countries’ representative agents have the same period utility so that 
\[
U_1 = u(C_1) + \beta u(C_2) \quad \text{and} \quad U_1^* = u(C_1^*) + \beta u(C_2^*),
\]
Assume period utility \( u(\cdot) \) to be isoelastic. International financial market clearing \( S_1 + S_1^* = I_1 + I_1^* \) (or \( CA_1 = -CA_1^* \)) determines the world interest rate \( r \).

1. State the intertemporal optimality conditions for production and show that an anticipated productivity shock \( dA_2/A_2 \) to production in Home changes equilibrium investment and the interest rate in the following way
\[
r \frac{dA_2}{A_2} + \left( \frac{\partial I_1}{\partial r} \right)^{-1} dI_1 - dr = 0.
\]
Derive the according equation for Foreign and \( dA_2^*/A_2^* \).

2. State the intertemporal optimality conditions for consumption and totally differentiate them with respect to \( dC_1, dC_1^*, dr, dA_2 \) and \( dA_2^* \). Use \( CA_1 = Y_1 - C_1 - I_1 \) and \( CA_1^* = -CA_1^* \) to restate the total derivatives in terms of \( dCA_1, dr, dA_2 \) and \( dA_2^* \).

3. Use the intertemporal optimality conditions for consumption, along with your results in 1 and 2, to show that an anticipated productivity shock \( dA_2/A_2 \) to production in Home changes the equilibrium current account level and the interest rate in the following way
\[
D \frac{dA_2}{A_2} - M \ dr + V dCA_1 = 0,
\]
for some functions \( V, D, \) and \( M \). Derive the according equation for Foreign and \( dA_2^*/A_2^* \) (with \( V^*, D^*, M^* \)) but using \( CA_1 = -CA_1^* \). Derive the signs of \( V, V^*, D \) and \( D^* \). On what do the signs of \( M \) and \( M^* \) depend?
4. Show that anticipated productivity shocks to Home and Foreign have the following effect on the equilibrium interest rate and the Home current account:

\[
\frac{d r}{d} = \frac{1}{M} \left( \frac{D}{V} \frac{d A_2}{A_2} + \frac{D^*}{V^*} \frac{d A^*_2}{A^*_2} \right)
\]

\[
dCA_1 = \frac{1}{V M} \left( \frac{D}{M} \frac{d A_2}{A_2} - \frac{D^*}{M^*} \frac{d A^*_2}{A^*_2} \right).
\]

Prove that \( dr > 0 \) for isoelastic utility.

5. Suppose Home runs a current account surplus \( CA_1 > 0 \) during period 1. How does an anticipated increase in Home productivity \( A_2 \) during period 2 affect the current accounts in period 1? Is the effect unambiguous? Why or why not? [Hint: You may want to use \( dr \) from 4 to derive \( dCA_1 \).]

6. Suppose again Home runs a current account surplus \( CA_1 > 0 \) during period 1. How does an anticipated increase in Foreign productivity \( A^*_2 \) during period 2 affect the current accounts in period 1? Is the effect unambiguous? Why or why not? [Hint: You may want to use \( dr \) from 4 to derive \( dCA_1 \).]

7. Consider anticipated and equal proportional increases in Home and Foreign productivity \( dA_2/A_2 = dA^*_2/A^*_2 \). How does this change affect worldwide investment \( I_1 + I_1^* \) during period 1? Does the answer depend on the elasticity of intertemporal substitution? Is the effect unambiguous? Why or why not?

2 Exponential Period Utility

There are two periods. A country’s representative household has the exponential period utility function

\[
u(C) = -\gamma \exp\left(-\frac{C}{\gamma}\right)
\]

with \( \gamma \in (0, \infty) \) and maximizes lifetime utility \( U_1 = u(C_1) + \beta u(C_2) \) subject to

\[
C_1 + RC_2 = Y_1 + RY_2 \equiv W,
\]

where \( R \equiv 1/(1+r) \) is the price of tomorrow’s consumption in terms of today’s consumption and \( W \) is initial wealth. The value of \( W \) depends on \( R \).

1. Derive the Euler equation and solve it for \( C_2 \) as a function of \( C_1 \), \( R \) and \( \beta \).

2. What is the optimal level of \( C_1 \) considering \( W \), \( R \) and \( \beta \) as given?

3. Differentiate this consumption function of \( C_1 \) with respect to \( R \) (differentiate \( W \) with respect to \( R \) too) and show that

\[
\frac{dC_1}{dR} = -\frac{C_1}{1+R} + \frac{Y_2}{1+R} + \frac{\gamma}{1+R} \left(1 - \ln(\beta/R)\right)
\]
4. Derive the intertemporal elasticity of substitution of the exponential period utility \((-u'(C)/Cu''(C))\).

5. Use this result to show that the derivative \(dC_1/dR\) in part 3 can be expressed as

\[
\frac{dC_1}{dR} = \frac{\sigma(C_2) C_2}{1 + R} - \frac{C_2}{1 + R} + \frac{Y_2}{1 + R}
\]

Interpret the three additive terms in this derivative.

3 Stochastic Current Account Model

There are infinitely many periods. A country’s representative household has the linear-quadratic period utility function

\[u(C) = C - \frac{a_0}{2} C^2\]

with \(a_0 \in (0, \infty)\) and maximizes lifetime utility

\[U_t = E_t \left[ \sum_{s=t}^{\infty} \beta^{s-t} u(C_s) \right]\]

subject to

\[CA_s = B_{s+1} - B_s = rB_s + \tilde{Z}_s - C_s \quad \forall s \geq t\]

where \(R \equiv 1/(1 + r) = \beta\) and \(\tilde{Z}_s(= \tilde{Y}_s - \tilde{G}_s - \tilde{I}_s)\) is random net output.

1. Derive the stochastic Euler equations and show that \(C_t\) satisfies

\[C_t = rR \left( (1 + r)B_t + \sum_{s=t}^{\infty} R^{s-t} E_t [\tilde{Z}_s] \right)\]

2. Show that \(CA_t \equiv B_{t+1} - B_t = \tilde{Z}_t - E_t [\tilde{Z}_t]\), where the hat denotes the permanent level of the variable. The permanent level \(\tilde{X}\) of a random variable \(X\) is defined as \(\sum_{s=t}^{\infty} R^{s-t} \tilde{X} = E_t \left[ \sum_{s=t}^{\infty} R^{s-t} \tilde{X}_s \right]\).

3. Define \(\Delta \tilde{Z}_s \equiv \tilde{Z}_{s+1} - \tilde{Z}_s\) and suppose that \(\lim_{T \to \infty} R^T E_t [\tilde{Z}_{t+T}] = 0\).

Show that the current account follows a martingale, that is: show that current account innovations (unexpected changes to the current account) are unrelated to any past realizations of state variables.

Hint: Show that the current account can be rewritten as

\[CA_t = -R \sum_{s=t}^{\infty} R^{s-t} E_t [\Delta \tilde{Z}_s]\]

for \(\lim_{T \to \infty} R^T E_t [\tilde{Z}_{t+T}] = 0\) and find \(CA_t - E_{t-1}[CA_t]\).

4. How is this finding related to Hall’s (1978) result that consumption follows a martingale?
4 Current Account and Terms of Trade

In a small open economy, the representative individual maximizes the lifetime utility function

\[ U_t = \sum_{s=t}^{\infty} \beta^{s-t} \left( \frac{X_{s}^{\gamma} M_{s}^{1-\gamma}}{P_{s}} \right)^{1-1/\sigma} - 1, \]

where \( X \) is consumption of an exported good and \( M \) consumption of an imported good. The country completely specializes in production of the export good. The endowment of this good is constant at \( Y \). The representative individual faces the fixed world interest rate \( r = (1 - \beta) / \beta \) in terms of the real consumption index \( C = X^{\gamma} M^{1-\gamma} \) (so a loan of 1 real consumption unit today returns 1 + \( r \) real consumption units tomorrow). There is no investment or government spending.

1. Let \( p \) be the price of the export goods in terms of the import good. So, a rise in \( p \) is an improvement in the terms of trade. Show that the consumption-based price index \( P \) in terms of imports is

\[ P = \frac{p^{\gamma}}{(1 - \gamma)^{1-\gamma}}. \]

2. Show that the home country’s current account identity is

\[ B_{t+1} - B_t = rB_t + \frac{p_t(Y - X_t)}{P_t} - \frac{M_t}{P_t}. \]

What is the corresponding intertemporal budget constraint for the representative consumer?

3. Show that utility maximization (Marshallian demands for \( X_t \) and \( M_t \)) and expenditure minimization (Hicksian demands for \( X_t \) and \( M_t \)) both imply that \( P_tC_t = p_tX_t + M_t \).

4. Derive the first-order conditions of the representative agent’s intertemporal consumption problem. What are the optimal paths for \( C_t, X_t \) and \( M_t \)? For this purpose, express \( C_t \) in terms of the representative agent’s present net wealth using the intertemporal budget constraint.

5. Suppose initial expectations are that \( p \) remains constant over time. There is an unexpected temporary fall in the terms of trade from \( p \) to \( p' < p \). What is the effect on the current account \( CA_t = B_{t+1} - B_t \) from part 2? What if \( p \) permanently drops to \( p' \)?

6. Now suppose foreign net wealth \( B \) is indexed to the import good \( M \) rather than to real consumption. Accordingly, let \( r \) denote the own-rate of interest on the import-denominated bond but assume again that \( r = (1 - \beta) / \beta \). How does a temporary drop in the terms of trade from \( p \) to \( p' < p \) affect the current account now? How do you explain differences, if any, to part 5? [Hint: You might find it instructive to consider the effect of a one-percent change in \( p_t \) on \( p_t/P_t \) and the current account balance under either denomination.]