1 Optimal Consumption with Complete and Incomplete Asset Markets

Consider a two-period model of consumption by a representative agent, who faces a market interest rate $r$ for riskless loans $B_2$. Labor earnings today are $Y_1$. There are $S$ states of nature tomorrow, and earnings realizations $Y_2(s)$ differ across states. Each state $s$ occurs with a probability $\pi(s)$. The representative consumer maximizes expected life-time utility

$$U_1 = C_1 - \frac{a_0}{2} (C_1)^2 + \beta E_1 \left[ C_2 - \frac{a_0}{2} (C_2)^2 \right],$$

where period utility is quadratic and $a_0 > 0$. The consumer’s time preference parameter $\beta$ is such that $\beta = \frac{1}{1+r}$. You may assume that $Y_1$ and all $Y_2(s)$ levels are small enough so that the marginal utility of period consumption $1 - a_0 C$ is strictly positive for all consumption levels.

When asset markets are incomplete, the relevant constraints can be written as

$$B_2 = (1+r)B_1 + Y_1 - C_1$$

$$C_2(s) = (1+r)B_2 + Y_2(s) \quad \forall s \in \{1, \ldots, S\}$$

Initial bond holdings $B_1$ are given and $B_2$ denotes bonds accumulated through the end of period 1.

1. Show that the preceding constraints imply the $S$ intertemporal budget constraints

$$C_1 + \frac{C_2(s)}{1+r} = (1+r)B_1 + Y_1 + \frac{Y_2(s)}{1+r}$$

for all states $s \in \{1, \ldots, S\}$.

2. Temporarily ignore the nonnegativity constraints $C_2(s) \geq 0$ for states $s$ of nature tomorrow. Compute the optimal level of consumption $C_1$ today. What are the implied values of $C_2(s)$. What would the optimal level of $C_1$ be for an infinitely-lived agent and output uncertainty in each future period?
3. Consider the nonnegativity constraint on $C_2$. Relabel the states of nature such that $Y_2(1) = \min_s \{Y_2(s)\}$. Show that if

$$(1 + r)B_1 + Y_1 + \frac{Y_2(1)}{1 + r} \geq E_1[Y_2]$$

then the $C_1$ computed in part 2 (for the two-period case) is still valid. What is the intuition? Suppose the preceding inequality fails to hold. Show that optimal consumption $C_1$ today is lower and equals

$$C_1 = (1 + r)B_1 + Y_1 + \frac{Y_2(1)}{1 + r}.$$ 

This is a precautionary savings effect. Explain why it arises. Does the bond Euler equation hold in this case?

[Hint: Apply the Kuhn-Tucker theorem to derive optimal consumption $C_1$.]

4. Assume the consumer faces complete global asset markets with $p(s)/(1+r)$, the state $s$ Arrow-Debreu security price, equal to $\pi(s)/(1 + r)$. Explain why these are called actuarially fair prices. Find the optimal values of $C_1$ and $C_2(s)$. Why can nonnegativity constraints be disregarded in the case of complete asset markets?

### 2 Consumption-based CAPM model (from Lucas, Econometrica 1978)

Consider a representative agent and a production process, in which a random and exogenous amount of perishable output $y_t$ falls from one fruit tree each period. There is no other output. The fruit output follows the stochastic process

$$\ln y_t = \ln y_{t-1} + \epsilon_t,$$

where $\epsilon_t$ is an unanticipated Gaussian shock with $E_{t-1}[\epsilon_t] = 0$ and $N(0, \sigma_2)$. So, $y_t$ has a lognormal distribution. There is no investment, that is there is no way to grow more fruit trees.

The agent’s life-time utility function takes the particular form

$$U_t = E_t \left[ \sum_{s=t}^{\infty} e^{-\theta(s-t)} u(c_s) \right],$$

where $\theta > 0$ is the rate of time preference. Assume there is a competitive stock market, in which people can trade shares in the fruit tree. The price of a share is denoted $p_t$ and ex-dividend. So, if the agent buys a share on date $t$, she gets her first dividend payment only on date $t + 1$.

1. Argue that a share holder’s individual budget constraint is $c_s + p_s x_{s+1} \leq (y_s + p_s)x_s$, where $x_s$ denotes the shares in the tree that the agent holds
at the end of period $s - 1$. Setup up the Bellman equation for the representative agent’s intertemporal consumption and share-holding choice. Show that the optimal consumption path satisfies the Euler equations

$$p_s u'(c_s) = e^{-\theta} E_s \left[ (p_{s+1} + y_{s+1}) u'(c_{s+1}) \right]$$  

(2-2)

for all $s \geq t$.

2. Show that, in equilibrium, the fundamental bubble-free price path of the shares in the tree is

$$p_t^* = E_t \left[ \sum_{s=t+1}^{\infty} e^{-\theta(s-t)} \frac{u'(y_s) y_s}{u'(y_t)} \right]$$

Interpret this formula in terms of expected payoffs and a “risk premium” by rewriting it in CAPM-style. What is the sign of the risk premium on the tree, and why?

3. Let agents have the iso-elastic period utility function

$$u(c) = \frac{c^{1-\rho} - 1}{1 - \rho},$$

where $\rho > 0$. Show that this utility function, together with the output process (2-1) implies

$$E_t \left[ (y_s)^{1-\rho} \right] = (y_t)^{1-\rho} e^{\sigma^2(1-\rho)^2(s-t)/2}.$$  

[Hint: You may find it helpful to use the moment generating function (mgf) for a normal probability distribution. The mgf of a normally distributed random variable $X$ (with $X \sim N(\mu, \sigma^2)$) is $M_X(t) \equiv E \left[ e^{-tX} \right] = e^{\mu t + \sigma^2 t^2/2}$.]

4. Deduce from your finding in part 3 that if $\theta > \sigma^2(1-\rho)^2/2$, then $p_t^* = \chi y_t$ for

$$\chi \equiv \frac{1}{e^{\theta^*(1-\rho)^2/2} - 1}.$$  

5. Return to a general, strictly concave period utility function $u(c)$. Let $b_t$ be the random variable $b_t = A(y_t)^{\lambda}/u'(y_t)$, where $\lambda \equiv \sqrt{2\theta}/\sigma^2$ and $A$ is an arbitrary constant. Show that $p_t^* + b_t$ will satisfy the individual’s Euler equation (recall part a) in equilibrium. So, $b_t$ is an asset price bubble. Show that $p_t = p_t^* + b_t$ violates the transversality condition

$$\lim_{T \to \infty} e^{-\theta(T-t)} E_t \left[ u'(y_{t+T}) p_{t+T} \right] = 0.$$  

(2-3)
6. Together with the equilibrium Euler equations

\[ p_s u'(c_s) = e^{-\theta E_t [(p_{s+1} + y_{s+1}) u'(c_{s+1})]} \]

from (2-2), the transversality condition (2-3) is sufficient for a stochastic price path \( \{p_s\}_{s=1}^{\infty} \) to be an equilibrium. You are asked to show here that (2-3) is also a necessary condition for an equilibrium. Iterate (2-2) forward to derive

\[ p_t u'(y_t) = E_t \left[ \sum_{s=t+1}^{\infty} e^{-\theta(s-t)} u'(y_s) y_s \right] + \lim_{T \to \infty} e^{-\theta(T-t)} E_t [u'(y_{t+T}) p_{t+T}] . \]

Argue that if the limit above is strictly negative, everyone would want to buy more of the tree and never sell it. Then argue conversely that it cannot be an equilibrium either if the limit is strictly positive. Why? [Hint: Look for a strategy that raises lifetime utility infinitely through the sale of a tiny fruit tree share today that is never repurchased.] Why does the Euler equation (2-2) not suffice to rule out an asset price bubble?

### 3 Cagan’s Monetary Model

Consider the following money-in-the-utility-function model in continuous time. Agents maximize

\[ \int_{t}^{\infty} e^{-\rho(s-t)} [u(c(s)) + v(m(s))] ds . \]

1. There are two types of assets, money \( M \) and a nominal bond \( B \). Total nominal asset holdings \( A \) are \( A = M + B \). The representative agent accumulates nominal assets at a rate \( \dot{A} = P(y - c) + iB \). Assume the Fisher parity holds and \( i = r + \pi \), where \( \pi = \dot{P}/P \) is the inflation rate. Show that the intertemporal budget constraint can be written in real terms as

\[ \dot{a} = (y - c) + ra - im , \]

where \( a \equiv A/P \), \( m \equiv M/P \) are real holdings of nominal assets.

2. Assume that

\[ v(m) = m^{\gamma}/\gamma , \]

where \( \gamma \in (-\infty, 1) \). [For \( \gamma = 0 \), \( v(m) = \ln m \).] Show that in this model, the elasticity of money demand with respect to the nominal interest rate \( i \) is

\[ -\frac{d \ln m}{d \ln i} = \frac{1}{1 - \gamma} . \]

[You may but need not use a Hamiltonian. An argument about marginal utility equalization to infer the first-order condition is fine.]
3. Argue that in equilibrium \( \rho = r \) and \( c = y \). Suppose nominal money supply \( M \) grows at a rate \( \frac{\dot{M}}{M} = \mu \). Derive the equation of motion for real money holdings

\[
\frac{\dot{m}}{m} = \mu + \rho - \frac{m^{\gamma-1}}{u'(y)}.
\]

4. Keep assuming that \( v(m) = m^\gamma / \gamma \), where \( \gamma \in (-\infty,1) \). Show that, in equilibrium and under constant money supply, speculative hyperinflations such that \( P \to \infty \) can arise only if the interest elasticity of money demand exceeds unity. [Hint: Speculative hyperinflations result in \( m = 0 \) in the limit. This is possible iff \( \lim_{m \to 0} mv'(m) = 0 \). Why?]

5. Cagan thought it more plausible that the interest elasticity of money, rather than being constant, rises as expected inflation rises. Assume that

\[
v(m) = \frac{m}{\gamma} \left[ 1 - \ln \left( \frac{m}{\kappa} \right) \right].
\]

(3-4)

for some \( \kappa > 0 \). Normalize output so that \( u'(y) = 1 \). Show that \( v'(m) > 0 \) for \( m < \kappa \) and that \( v'' < 0 \). Show that money demand is given by

\[m = \kappa e^{-\gamma i},\]

the so-called Cagan equation.

Verify that, for this equation, the interest elasticity of money demand is \( \gamma i \), which tends to infinity as \( i \to \infty \).

6. Show that in the Cagan version of utility from money holdings (3-4)

\[
\lim_{m \to 0} mv'(m) = 0
\]

so that speculative hyperinflations are possible. [Hint: Invoke L'Hôpital's Rule.]

7. Assume there is a fixed flow of government spending \( g \), which is not tax funded but financed by money creation instead. The central bank monetizes the government deficit \( d^{CB} \). So, the government’s budget constraint is

\[g = d^{CB} = \frac{\dot{M}(s)}{P(s)} = \frac{\dot{M}(s)}{M(s)}m(s) = \mu(s)m(s),\]

where the rate of nominal money supply growth \( \mu(s) \) is an endogenous variable. Assume money demand is of the Cagan form

\[m = \kappa e^{-\gamma i},\]

where \( i(s) = \rho + \pi(s) \). Using the equilibrium condition that the inflation rate equals money supply growth less money demand growth,

\[\pi(s) = \mu(s) - \frac{\dot{m}(s)}{m(s)},\]
derive a differential equation of the form \( \dot{\pi}(s) = f(\pi(s), q) \) that characterizes the equilibrium. Graph \( \dot{\pi} \) on the vertical axis against \( \pi \) on the horizontal axis and show that there can be two different steady-state inflation rates. Also show that the low-inflation steady state is dynamically unstable and that the high-inflation steady state is dynamically stable.