1 Productivity Shocks with Initially Unbalanced Current Accounts

There are two periods and two countries. Home produces with \( Y = AF(K) \) and Foreign with \( Y^* = A^*F(K^*) \), where \( A \) and \( A^* \) are productivity parameters. Both countries’ representative agents have the same period utility so that \( U_1 = u(C_1) + \beta u(C_2) \) and \( U_1^* = u(C_1^*) + \beta u(C_2^*) \). Assume period utility \( u(\cdot) \) to be \textit{isoelastic}. International financial market clearing \( S + S^* = I + I^* \) (or \( CA_1 = -CA_1^* \)) determines the world interest rate \( r \).

1. Use the intertemporal optimality conditions for production to show that an anticipated productivity shock \( dA_2 / A_2 \) to production in Home changes equilibrium investment and the interest rate in the following way

\[
r \frac{dA_2}{A_2} + \left( \frac{\partial I_1}{\partial r} \right)^{-1} dI_1 - dr = 0.
\]

Derive the according equation for Foreign and \( dA_2^* / A_2^* \).

2. Use the intertemporal optimality conditions for consumption, along with your results in 1, to show that an anticipated productivity shock \( dA_2 / A_2 \) to production in Home changes the equilibrium current account level and the interest rate in the following way

\[
D \frac{dA_2}{A_2} - M dr + V dCA_1 = 0,
\]

for some functions \( V, D, \) and \( M \). Derive the according equation for Foreign and \( dA_2^* / A_2^* \) (with \( V^*, D^*, M^* \) and using \( CA_1 = -CA_1^* \)). Derive the signs of \( V, V^*, D \) and \( D^* \).

3. Show that anticipated productivity shocks to Home and Foreign have the following effect on the equilibrium interest rate and the Home current account:

\[
\frac{dr}{dA_2} = -\frac{1}{\frac{M}{V} + \frac{M^*}{V^*}} \left( \frac{D}{V} \frac{dA_2}{A_2} + \frac{D^*}{V^*} \frac{dA_2^*}{A_2^*} \right)
\]

\[
\frac{dCA_1}{dA_2} = \frac{1}{\frac{M}{V} + \frac{M^*}{V^*}} \left( \frac{D}{M} \frac{dA_2}{A_2} - \frac{D^*}{M^*} \frac{dA_2^*}{A_2^*} \right).
\]
4. Suppose Home runs a current account surplus \( CA_1 > 0 \) during period 1. How does an anticipated increase in Home productivity \( A_2 \) during period 2 affect the current accounts in period 1? Is the effect unambiguous? Why or why not?

5. Suppose again Home runs a current account surplus \( CA_1 > 0 \) during period 1. How does an anticipated increase in Foreign productivity \( A_2^* \) during period 2 affect the current accounts in period 1? Is the effect unambiguous? Why or why not?

6. Consider anticipated and equal proportional increases in Home and Foreign productivity \( dA_2 / A_2 = dA_2^* / A_2^* \). How does this change affect worldwide investment \( I_1 + I_1^* \) during period 1? Does the answer depend on the elasticity of intertemporal substitution? Is the effect unambiguous? Why or why not?

2 Exponential Period Utility

There are two periods. A country’s representative household has the exponential period utility function

\[
u(C) = -\gamma \exp(-C/\gamma)\]

with \( \gamma \in (0, \infty) \) and maximizes lifetime utility \( U_1 = u(C_1) + \beta u(C_2) \) subject to

\[C_1 + RC_2 = Y_1 + RY_2 \equiv W,\]

where \( R \equiv 1/(1 + r) \) is the price of tomorrow’s consumption in terms of today’s consumption and \( W \) is initial wealth. The value of \( W \) depends on \( R \).

1. Derive the Euler equation and solve it for \( C_2 \) as a function of \( C_1, R \) and \( \beta \).

2. What is the optimal level of \( C_1 \) considering \( W, R \) and \( \beta \) as given?

3. Differentiate this consumption function of \( C_1 \) with respect to \( R \) (differentiate \( W \) with respect to \( R \) too) and show that

\[
\frac{dC_1}{dR} = -\frac{C_1}{1 + R} + \frac{Y_2}{1 + R} + \frac{\gamma}{1 + R} (1 - \ln(\beta/R))
\]

4. Derive the intertemporal elasticity of substitution of the exponential period utility \( -u'(C)/Cu''(C) \).

5. Use this result to show that the derivative \( dC_1/dR \) in part 3 can be expressed as

\[
\frac{dC_1}{dR} = \frac{\sigma(C_2) C_2}{1 + R} - \frac{C_2}{1 + R} + \frac{Y_2}{1 + R}.
\]

Interpret the three additive terms in this derivative.
3 Stochastic Current Account Model

There are infinitely many periods. A country’s representative household has the linear-quadratic period utility function

\[ u(C) = C - \frac{a_0}{2} C^2 \]

with \( a_0 \in (0, \infty) \) and maximizes lifetime utility

\[ U_t = \mathbb{E}_t \left[ \sum_{s=t}^{\infty} \beta^{s-t} u(C_s) \right] \]

subject to

\[ CA_s = B_{s+1} - B_s = rB_s + \tilde{Z}_s - C_s \quad \forall s \geq t \]

where \( R \equiv 1/(1 + r) = \beta \) and \( \tilde{Z}_s(\equiv \tilde{Y}_s - \tilde{G}_s - \tilde{I}_s) \) is random net output.

1. Derive the stochastic Euler equations and show that \( C_t \) satisfies

\[ C_t = rR \left( (1 + r)B_t + \sum_{s=t}^{\infty} R^{s-t} \mathbb{E}_t[\tilde{Z}_s] \right) \]

2. Show that then \( CA_t \equiv B_{t+1} - B_t = \tilde{Z}_t - \mathbb{E}_t[\tilde{Z}_t] \), where the hat denotes the permanent level of the variable. The permanent level \( \bar{X} \) of a random variable \( X \) is defined as \( \sum_{s=t}^{\infty} R^{s-t} \bar{X} \equiv \mathbb{E}_t \left[ \sum_{s=t}^{\infty} R^{s-t} \tilde{X}_s \right] \).

3. Define \( \Delta \tilde{Z}_s \equiv \tilde{Z}_s - \tilde{Z}_{s-1} \) and show that the current account can be rewritten as \( CA_t = -\sum_{s=t+1}^{\infty} R^{s-t}\mathbb{E}_t[\Delta \tilde{Z}_s] \).

4. Use \( \Delta \tilde{Z}_t \equiv \tilde{Z}_t - \tilde{Z}_{t-1} \) to show that the quantity

\[ CA_t - (1 + r)CA_{t-1} - \Delta \tilde{Z}_t \]

is stochastically independent of \( CA_s \) and \( \Delta \tilde{Z}_s \) for all \( s < t \). How is this finding related to Hall’s (1978) result that consumption follows a random walk?

4 Current Account and Terms of Trade

In a small open economy, the representative individual maximizes the lifetime utility function

\[ U_t = \sum_{s=t}^{\infty} \beta^{s-t} \left( \frac{X_s^\gamma M_s^{1-\gamma}}{1 - \frac{1}{\sigma}} \right)^{1-1/\sigma} - 1 \]

where \( X \) is consumption of an exported good and \( M \) consumption of an imported good. The country completely specializes in production of the export good. The
endowment of this good is constant at $Y$. The representative individual faces the fixed world interest rate $r = (1 - \beta)/\beta$ in terms of the real consumption index $C = X^\gamma M^{1-\gamma}$ (so a loan of 1 real consumption unit today returns $1 + r$ real consumption units tomorrow). There is no investment or government spending.

1. Let $p$ bet the price of the export goods in terms of the import good. So, a rise in $p$ is an improvement in the terms of trade. Show that the consumption-based price index $P$ in terms of imports is

$$P = p^\gamma / \gamma^\gamma (1 - \gamma)^{1-\gamma}.$$ 

2. Show that the home country’s current account identity is

$$B_{t+1} - B_t = rB_t + \frac{p_t(Y - X_t)}{P_t} - \frac{M_t}{P_t}.$$ 

What is the corresponding intertemporal budget constraint for the representative consumer?

3. Derive the first-order conditions of the consumer’s problem. (Hint: Reformulate the utility function and budget constraint in terms of real consumption $C$.) What are the optimal paths for $X$ and $M$?

4. Suppose initial expectations are that $p$ remains constant over time. There is an unexpected temporary fall in the terms of trade from $p$ to $p' < p$. What is the effect on the current account $CA_t = B_{t+1} - B_t$ from part 2?

5. Now suppose foreign net wealth $B$ is indexed to the import good $M$ rather than to real consumption. Accordingly, let $r$ denote the own-rate of interest in imports but assume again that $r = (1 - \beta)/\beta$. How does a temporary drop in the terms of trade from $p$ to $p' < p$ affect the current account now? How do you explain differences, if any, to part 4?