Piketty's *Capital in the Twenty-First Century* under the Lens of a Simple Economic Model[*]

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Under what conditions can we expect Piketty's main two tenets to be right, namely that $r > g$ and that β is high and might rise further? What do these conditions imply for growth and the distribution of income between capital and labor? The first tenet $r > g$ states that the real rate of return on physical capital exceeds the economy-wide growth rate of output. The second tenet holds that the capital-output ratio $\beta = K/Q$ is now back at its historic peak level of between 6 and 7, and the upward trend may continue.

I will consider some basic (mainstream growth) economics. Call the setup "neoclassical" if you wish because only the most basic microeconomics assumptions will be allowed.

Assumption 1 *A country's aggregate production function combines physical capital K and labor L under constant returns to scale and a total factor productivity A into aggregate output*

$$Q = A \cdot F(K, L). \tag{1}$$

Importantly, the production factors K and L can be complements or substitutes. I will place no restriction on those possibilities. The rate of substitution between capital and labor may in fact change—an outcome that Piketty explores himself at some length. In fact, a few of the scenarios that I consider are particularly plausible if the rate of substitution between capital and labor moves over time, but none of my arguments will require a particular assumption on that substitutability.

There is no human capital, as is Piketty's preference in *Capital in the Twenty-First Century*. Labor L is therefore the simple head count of a country's (actively working) population. However, the following derivations can easily be generalized to a composite of several other factors of production beyond physical capital K. A single other factor L just keeps the exposition straight (I will consider some human capital when helpful). The multiplicative total factor productivity term A is a convenient scalar in front of the production function, but none of the following derivations depend on whether A augments only one factor or the other, or both, or production as a whole.

Assumption 2 *There are constant returns to scale in production.*

If you like it formal, constant returns to scale mean that the production function $F(K, L)$ is homogeneous of degree one in its arguments. The following derivations will be conditioned on constant returns to scale, mostly to enhance clarity of the basic mechanisms. Increasing returns to scale are in most cases just an extension. A downside of an extension to increasing returns to scale would be that the additional output proceeds from scale economies need to be arbitrarily assigned to one factor of production or the other, taking away from the clarity of exposition under constant returns to scale, where the forces of free markets assign the incomes.

Under constant returns to scale (Assumption 2), the production function in Assumption 1 can be restated in per-capita terms:

$$q = \frac{Q}{L} = A \cdot F(k, 1) \equiv Af(k), \tag{2}$$

where $q \equiv Q/L$ is per-capita output (the output-labor ratio), $k \equiv K/L$ is the capital-labor ratio and arguably the most crucial variable to making sense of Piketty. The lower-case production function $f(k)$ is defined as $f(k) \equiv F(k, 1)$ and useful in stating most conditions as compactly as possible.

The value of GDP is $Y = P \cdot Q$, where P is the consumer price index of the economy's basket of products. The price level P plays no relevant role in the long term of a century (it plays no role in a microeconomically founded theory of growth). We can therefore just follow Milton Friedman's grandiose statement that "inflation is always and everywhere a monetary phenomenon" and assume with no loss of generality that money growth is such that $P = 1$ at all times throughout the century. Then GDP and real output are the same, $Y = Q$, and all factor prices are essentially relative prices in terms of the value of the economy's basket of output. The real interest rate r/P and the real wage w/P just become r and w for $P = 1$.

(I am aware that Piketty argues that inflation does have considerable distributional consequences. That is true. Inflation essentially wipes out nominal debt, because debt contracts are typically written with no regard to inflation, so high inflation eradicates the wealth of savers, lenders and owners of financial capital. There are at least three reasons to put that perspective aside. First, business owners and owners of physical capital are unaffected, because the business and physical capital retain their real value. Second, following Milton Friedman tenet that "inflation is always and everywhere a monetary phenomenon," a societey can always choose to instigate a hyper-inflation merely through excessive money printing, alleviate all debtors of their burden, and redistribute financial wealth that way if desired. Third, we could simply start writing debt contracts that correct for inflation and specify the real terms of repayment, and all distributional consequences of inflation would be gone. Those are all choices that do not have much to do with the long-term growth trajectory of an economy and the earnings of its factors of production. Piketty himself writes: "The effect [of inflation] is largely one of redistributing wealth among asset categories rather than a long-term structural effect.")

The real return on physical capital r**.** Production without waste (that is profit maximization of the capitalist firms) implies the two so-called input rules for optimal capital use and labor employment:

$$
\begin{aligned}
r &= A\, f'(k) \\
w &= q - r\, k = A\, f(k) - A\, f'(k)\, k.
\end{aligned}
$$

The first input rule states that producers stop adding additional capital per worker k once the incremental (marginal) product of an additional unit of capital per worker $Af'(k)$ has fallen to the level of the real interest rate. Beyond that point of optimality, adding one more unit of capital per worker would generate smaller revenues than it costs ($Af'(k) < r$), so capitalists optimally stop adding capital per worker as soon as $r = Af'(k)$. (Constant returns to scale under Assumption 2 imply that $f'(k)$ falls and falls as capital use per worker increases, but the marginal product of capital per worker is always positive $Af'(k) > 0$. Piketty perhaps slightly misrepresents the issue when arguing that the capital-income share $\beta = k/q$ determines the real interest rate r. It is k that matters.)

The second input rule states that workers earn the remaining revenues in the form of wages w, after capitalists have received their return on physical capital r. In general, all output generated in an economy must be someone's real income. When there are only two factors of production, then all the revenue that does not go to one factor of production as income must end up in the pockets of the other factor of production. Concretely, real revenues per worker are $q = Af(k)$. A per-worker unit of capital earns $r\,k = A\,f'(k)\,k$, which is the income of the capital owners. The left-over from revenues after the capitalists have been paid goes to workers. That remainder per worker is $q - r\,k$ and goes to each worker as a wage, so the second input rule follows.

The real growth rate of output g and the distribution of income. Growth rates are percentage numbers. We could denote the percentage change in output with $\hat{Q}$. Piketty prefers a single letter for the growth rate of real output and defines $g \equiv \hat{Q}$.

Two more definitions play a prominent role in Piketty's book. Piketty calls the capital-output ratio β and defines

$$\beta \equiv K/Q.$$

This is an important variable for the share of aggregate income that goes to capital owners. Piketty much refers to the fraction α of capital income in an economy's total income. That fraction α is a commonly used one and is widely defined as

$$\alpha \equiv rK/PQ,$$

not just by Piketty. The share of capital in national income α is the main measure of inequality in Piketty's book. An increasing income share of capital, that is a rise in α, means more income inequality, which ultimately translates into wealth inequality.[1] (Cobb and Douglas, when they

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[1] To see this rigorously, consider the variance of incomes, which is a basic measure of inequality (the variance has the relevant properties in common with any other good measure of inequality, see Shorrocks 1980, 1984). In a stylized society like Piketty's, there are two groups of people: N capitalists earning $rK/N = \alpha Q/N$ per person and L workers earning $w = wL/L = (1 - \alpha)Q/L$ per person. Say a member of society i earns an individual income y_i. Then mean income, or per-capita income, in this two-group society is simply $\mathbb{E}[y_i] = Q/(N + L)$. The variance of incomes in this two-group society of only capitalists and workers is equal to

$$\mathbb{V}(y_i) = \mathbb{E}[y_i^2] - \mathbb{E}[y_i]^2 = \left[\frac{\alpha^2}{N/(N+L)} + \frac{(1-\alpha)^2}{L/(N+L)} - 1\right]\frac{Q^2}{(N+L)^2}.$$

This variance only depends on α for constant sizes of the two societal groups. In fact, it is easy to show that the variance of incomes (inequality) increases if and only if $\alpha > N/(N+L)$, that is if capital owners earn more than their head-count share of national income. This is the case in reality.

wrote down an early version of a production function in the 1920s, thought of α as largely constant; for many a country and most time periods one-third is a widely accepted rough guess for α. That is by no means universal. For Brazil, for example, a fifty-fifty division of value added between labor and capital is the recent norm.)

In summary, Piketty's analysis is largely based on the following three relationships

$$
\begin{aligned}
g &\equiv \hat{Q}, \\
\beta &\equiv \frac{K}{Q}, \\
\alpha &\equiv \frac{rK}{PQ} = r\,\beta.
\end{aligned}
$$

(As an aside, in terms of our earlier production function we can also state $\alpha \equiv f'(k)\,k/f(k)$.)

Minimalist accounting.   How does the real interest rate r relate to the income share of capital α in this basic and thus very general setup? Piketty argues that the "natural" capital-output ratio β has recently returned to roughly 6 to 7 (600 to 700 percent) and that it may now keep trending upwards. The problem with all data is that they end in the present, so Piketty's second tenet is no more than a good guess. I will give Piketty the benefit of the doubt and assume his prediction is correct. Most pension funds will tell you that, in the long term, you should not expect a real rate of return r of more than 6 or 7 percent. That means, a simple guess for $\alpha = r\beta$ is somewhere between 36 percent ($.06 \times 600$) and 49 percent ($.07 \times 700$), clearly much higher than the conventional capital share in income of one-third.

Somewhat less minimalist accounting and Piketty's "natural" rule $r > g$.   Relating r and g to each other takes a little thought. At least to me, their relationship with inequality is not obvious.

Does $r > g$ aggravate inequality? It is a funny economic convention about returns to capital that we measure them in percentages, rather than as an absolute annual income per capital owner. The reason is probably that we like to count capital in units of money K, not in head counts of capitalists. A side effect is that both the interest rate r and the growth rate g are therefore percentage numbers, and Piketty can compare the two to each other. But that does not mean the comparison is informative. Think of the way we quote the returns to labor: we report an average wage per worker w in dollars per person (quite trivially the wage bill wL divided by the number of workers is $w = wL/L$). That's not what we do with the returns to capital. If we wanted to know the capital income per capitalist, we could simply count up the number of capital owners N, take their total capital income rK and find the capital income per capitalist: rK/N. The units of this measure are dollars per person, just as with wages. But that is not what we usually report. Instead we think of an interest rate, which measures the amount of dollar earnings per dollar of invested capital, so the interest rate has no unit (it is dollars per dollars) and looks as if it were comparable to a growth rate. Why not adopt a similar convention for labor income?

Suppose for a minute people were numbers, too. I know it may be offensive, but just for the sake of clarifying what r means. Suppose you compute the value of a person the same way as Piketty and his co-authors infer the value of assets in their empirical work. Income tax returns of the capitalists do not necessarily state the amount of invested capital, so Piketty's empirical approach

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is to use rK, combined with what he knows about typical returns for certain asset classes, and to infer backwards the value of the invested capital K. Concretely, one can take the declared income from a tax record, use a well established financial adjustment factor, and compute the value of the asset that generates the income. That's how Piketty and his co-authors infer values of physical capital K from income tax records. We could apply a similar idea to figure out wage income and the value of the asset that generates the wages: human capital H. Use the same established financial adjustment factors as Piketty does to infer physical capital K, but combine the wages with it, and that will tell you the value of our human capital H. (I am aware that Piketty dismisses human capital as unimportant but I cannot see what, in principle, would preclude the analysis given the general methodology.) The economic value of an average wage earning person is then the ratio H/L; so much for the offensive part. Under our new hypothetical convention we would say that tax returns show the wage bill ωH, and the reported return on human capital ω would now be a percentage number (while the wage per worker would be $w = \omega H/L$ but we would not use it). There is no other factor of production beyond physical capital and human capital, so all earnings in the economy must come from its production $rK + \omega H = Q$, and ω is now quoted in percent just like r, whereas K and H are both quoted in dollars.

We could compare the percentage returns on human capital and the percentage returns on physical capital to each other, and to the economy-wide growth rate. What if both $\omega > g$ and $r > g$, could we infer anything about the evolution of overall inequality? Maybe, maybe not. What if $\omega > r > g$? We don't know ω because we usually do not care to compute it, but it seems we would need to know. My point is simply that $r > g$ alone tells us little about the projected evolution of inequality. In particular, if $K/Q > 1$ (which Piketty documents being 6 to 7 now) and also $H/Q > 1$, then both $r > g$ and $\omega > g$ is possible; $r > \omega > g$ is possible, $\omega > r > g$ is possible as well. I do not see how either of the two latter conditions would be enough to infer whether inequality between capital and labor changes. We either need to bring in more information on ω in some form, or we need to do some more theory.

Let's do some more theory. More theory about the "natural" rule $r > g$. For our basic production function we can discern three possible sources of growth. The first source of growth is productivity change $\hat{A}$ and it is arguably the most lasting source of growth over the past centuries. $\hat{A}$ is the percentage change of total factor productivity A. A plausible long-term guess from looking back in history over the 20th century is that perpetual productivity growth keeps propelling the global economy at a long-term rate of 3 percent or so ($\hat{A} = .03$).

The second source of growth is capital deepening $\hat{k}$: an increase in k means that workers get matched with more equipment to help them produce. $\hat{k}$ is the percentage change in the capital-labor ratio $k = K/L$.

The third source of growth of total output is simply growth of the working population $\hat{L}$: if every worker produces as much output per person as before, using given productivity and the given physical capital per worker, output grows one-for-one with the size of the working population.

Under Assumptions 1 and 2, the relationship between the aggregate growth rate g and the three

individual sources of growth is:[2]

$$
\begin{aligned}
g &= \hat{A} + \alpha\,\hat{k} + \hat{L} \\
&= \hat{A} + r\beta\,\hat{k} + \hat{L}.
\end{aligned}
\tag{3}
$$

Two sources of growth translate into output change one-for-one. A one-percent productivity increase ($\hat{A} = .01$) raises the real GDP growth rate by one percentage point, and a one-percent rate of working population growth ($\hat{L} = .01$) raises the real GDP growth rate by one percentage point. In contrast, capital deepening $\hat{k}$, that is an increase in the ratio of physical equipment per worker k, translates into real GDP growth only by a fraction $\alpha < 1$. Under constant returns to scale (Assumption 2), pairing up workers with more equipment does raise the workers' production but the boost is mitigated as additional units of equipment help less and less. Additional equipment units always do help but equipment accumulation per worker runs into diminishing returns. (Under increasing returns to scale, it would be possible that capital accumulation translates into more than proportional growth in principle but that case does not appear to me to be empirically relevant.)

Equation (3) provides a rigorous and general relationship between the real rate of return on physical capital r and real output growth g. The relationship is based on only two assumptions: Production happens by combining physical capital with at least another factor of production (Assumption 1), and the contribution of capital deepening to output growth is less than one-for-one (in our setup a consequence of constant returns to scale under Assumption 2).[3] There are many possibilities, but anything does not go under Piketty's two tenets.

Why is $r > g$ so special? By Piketty's own account, a main measure of inequality from capital accumulation is α. An increase in α means that the share of capital in output is rising. By the mere definition of $\alpha = r\beta$, the proportional change in α is

$$
\hat{\alpha} = \hat{r} + \hat{\beta} = (\hat{r} - g) + \hat{k} + \hat{L},
$$

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[2]For a derivation, take logs of equation (1) to find $\ln Q = \ln A + \ln F(K, L)$. Then take the total derivative:

$$
\begin{aligned}
\hat{Q} = \frac{\mathrm{d}Q}{Q} &= \frac{\mathrm{d}A}{A} + \frac{[\partial F(\cdot,\cdot)/\partial K]\,K}{F(\cdot,\cdot)}\frac{\mathrm{d}K}{K} + \frac{[\partial F(\cdot,\cdot)/\partial L]\,L}{F(\cdot,\cdot)}\frac{\mathrm{d}L}{L} \\
&= \hat{A} + \frac{r\,k}{q}\hat{K} + \frac{w}{q}\hat{L}.
\end{aligned}
$$

for $q \equiv Q/L$. The last step makes use of the general input rules $r = Af'(k) = A[\partial F(\cdot,\cdot)/\partial K]$ and $w = A[\partial F(\cdot,\cdot)/\partial L]$, independent of whether there are constant returns to scale or not. Under constant returns to scale, the relationship simplifies further because $rk = \alpha q$ and $w = q - rk = (1-\alpha)q$:

$$
g = \hat{Q} = \hat{A} + \alpha\hat{K} + (1-\alpha)\hat{L}.
$$

(As you can see, however, increasing returns would work, too, the shares α and $(1-\alpha)$ would simply be scaled up by a factor of proportionality that reflects the scale economies; we would not be able to infer, however, which factor of production would ultimately get to pocket those gains from scale economies.) Finally, using the fact that $\hat{k} = \hat{K} - \hat{L}$ (by the mere definition of $k \equiv K/L$) in the above expression yields equation (3).

[3]More generally, a less than one-for-one contribution of capital deepening to growth is an arguably realistic consequence also of increasing returns to scale if those scale economies are not stronger than a factor of $1/\alpha$, say not larger than 300 percent ($1/.\bar{3}$).

where the second equality follows since the change in the capital-output ratio $\hat{\beta} = \hat{K} - g = \hat{k} + \hat{L} - g$ by the definitions $\beta = K/Q$ and $k = K/L$.

Let's pause for a moment. What truly matters for whether or not α-driven inequality increases, that is whether or not $\hat{\alpha}$ is positive or negative, is the difference between the proportional *change* in the interest rate r and the output growth rate g. I fail to see why $r > g$ should matter for the change in inequality, that is the change in α. The simple math of the basic and general model doesn't deliver a condition such as $r > g$, but instead a condition on $\hat{r}$. Let's nevertheless keep Piketty's first tenet $r > g$ in mind, and have it imposed on our economy. However, it really will be Piketty's second tenet that disciplines inequality and growth.

Let's study the prospective evolution of the real interest rate r and whether it will increase ($\hat{r} > 0$) or fall ($\hat{r} < 0$). Anything is possible in principle, but it turns out Piketty's projections for β place a lot of discipline on what can go on.

What about β?   As it turns out, under Piketty's second but less prominent tenet that a natural level of the capital-to-output ratio β is currently around 6 or 7, and that it might continue to grow, we have a range of permissible scenarios for inequality and they may not look so bad. By its definition, a constant or increasing capital-output ratio $\beta = K/Q$ of roughly 6 or 7 and rising means that

$$\hat{\beta} = \hat{K} - g = -g + \hat{k} + \hat{L} \geq 0,$$

because $\hat{k} = \hat{K} - \hat{L}$. Logically equivalently, if β remains constant or rises, then it must also be true that

$$\hat{k} \geq g - \hat{L}.$$

The relationship says that the rate of capital deepening must be at least equal to, or larger than, the economy-wide growth rate per capita, then and only then is Piketty's second tenet true that the capital-to-output ratio β remains constant or rises. For a Chinese-style growth rate of ten percent per year ($g = .10$) and no population growth ($\hat{L} = 0$), capital deepening $\hat{k}$ would have to be more than 10 percent per year. Say world output grows at a rate of five percent ($g = .05$) and the working population increases at a rate of 2 percent ($\hat{L} = .02$), then capital deepening must be at least as fast as 3 percent ($\hat{k} = .05 - .02 = .03$). Pick your numbers, and you can compute your very own favorite scenario that satisfies your own projections while still being true to Piketty (with $r > g$ and $\hat{\beta} \geq 0$).

We can refine the conditions. Recall that $g = \hat{A} + r\beta\hat{k} + \hat{L}$ by equation (3), so we can also write $\hat{k} \geq g - \hat{L} = \hat{A} + r\beta\hat{k}$, so

$$\hat{k} \geq \hat{A} + r\beta\hat{k}$$

and, equivalently,[4]

$$\hat{k} \geq \frac{\hat{A}}{1 - r\beta}.$$

This is an important point: Piketty's dark scenario that productivity growth $\hat{A}$ slows down to a negligible rate close to zero would actually imply that $\hat{k}$ slows down to zero, then an output

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[4]You may wonder whether the denominator $1 - r\beta$ is really positive (if it were negative then the inequality would actually be the other way). Empirically, there is little reason yet to doubt that $1 - r\beta$ is positive because $r < 1/\beta$ at an interest rate of 7 or 8 percent and $1/\beta$ is no less than 12 percent even at a β of 8.

growth rate equal to worker population growth $g = \hat{L}$ would result, and α would remain constant. Inequality would stop changing, in contradiction to Piketty's projection that inequality worsens. Let's keep a positive productivity growth $\hat{A} > 0$ as a main projection; it allows Piketty to be right about worsening inequality, but it also keeps propelling incomes for everyone—something we might be even more concerned with than inequality.

Let's return to the crucial condition for a change in inequality

$$\hat{\alpha} = \hat{r} + \hat{\beta} = (\hat{r} - g) + \hat{k} + \hat{L}$$

and use again the fact that $g = \hat{A} + r\beta\,\hat{k} + \hat{L}$ by equation (3). Substituting out for g, we find a logically necessary relationship between the change in the income share of capital and other changing variables.

$$\hat{\alpha} = \hat{r} + (1 - r\beta)\hat{k} - \hat{A}.$$

As mentioned, Piketty's restriction on the growth of β disciplines the possibilities quite vehemently. We just found that $\hat{k} \geq \hat{A}/[1 - r\beta]$ under Piketty's projection that the capital-to-output ratio β will be at least constant or rise further (his second tenet). Use that inequality on capital deepening in the preceding equation, then:

$$\hat{\alpha} \geq \hat{r} + (1 - r\beta)\frac{\hat{A}}{1 - r\beta} - \hat{A} = \hat{r}.$$

We've come a long way to see that the matters are actually quite simple under Piketty's two tenets. After all is said and done, Piketty's world comes down to just this: $\hat{\alpha} \geq \hat{r}$ (but the level of r does not matter).

Inequality will change no less than the interest rate *changes*. Again, the level of the interest rate r is actually irrelevant, so there goes the importance of the first tenet. Will inequality necessarily increase under Piketty's scenario (his second tenet) that β stays constant at 6 to 7, or rises further? No. Nothing about the economics of growth says that the two Piketty conditions necessarily result in an increase in inequality. The share of capital in economy-wide income $\alpha = rK/Q$ might actually fall when $r > g$ and β rises. Why? Recall that $\hat{r}$ can be negative. If you were to ask around among economists what a positive rate of capital deepening $\hat{k}$ implies for the interest rate r, all else equal, the probable answer would be: the real interest rate must be falling. As the economy accumulates more equipment per worker, it raises output but it also runs into diminishing returns to equipment, so the incremental output from any additional unit of capital per worker is less and less as we deepen capital use, and the real return on capital must be falling, all else equal.

Not all else is equal with Piketty, however, because imposing that β remains constant or rises disciplines the economy. A little more formally, the input rule says $r = Af'(k)$, so the proportional change in the real interest rate is

$$\hat{r} = \hat{A} + \rho(k)\hat{k},$$

where $\rho(k) \equiv Af''(k)/f'(k)$ is the elasticity of per-capita output with respect to the capital-labor ratio. This elasticity is closely related to the rate of substitution between capital and labor, which Piketty himself acknowledges as important. Arguably, the crucial factor for the evolution of inequality between capital and labor is the behavior of $\rho(k)$: the conditions $r > g$ and on β alone do not suffice to explain inequality change, so they alone cannot guide us looking back in history

or forward into the 21st century. We may not know much about $\rho(k)$ but one thing is quite sure: it must be negative. As we accumulate more equipment per worker ("deepening capital use"), we do raise output but we run into diminishing returns so the incremental output from any additional unit of capital per worker is less and less. It now depends on the empirical circumstances in the economy whether the interest rate is falling ($\hat{r} < 0$) or increasing ($\hat{r} > 0$).

Use the above condition that $\hat{k} \geq \hat{A}/[1 - r\beta]$ (following directly from Piketty's second tenet that the capital-to-output ratio β remains at least constant but may rise further), plug it into $\hat{r} = \hat{A} + \rho(k)\hat{k}$ and:

$$\hat{r} = \hat{A} + \rho(k)\hat{k} \leq \hat{A} + \rho(k)\frac{\hat{A}}{1 - r\beta} = \frac{1 - r\beta + \rho(k)}{1 - r\beta}\hat{A}.$$

This inequality follows inevitably from Piketty's scenario, and it is important. The inequality states that the change in the real interest rate is actually bounded from *above* in Pikett's projections. The real interest cannot rise any faster than $\hat{r} \leq \{[1 - r\beta + \rho(k)]/[1 - r\beta]\}\hat{A}$. And for a sufficiently negative $\rho(k)$, in particular for $\rho(k) < -(1 - r\beta)$, that upper bound is actually a negative number. Piketty writes on the issue of a potentially falling interest rate himself that "[t]oo much capital kills the return on capital. ... The interesting question is therefore not whether the marginal productivity of capital decreases when the stock of capital increases (this is obvious [*my remark*: it actually isn't as we just saw]) but rather how fast it decreases. In particular, the central question is how much the return on capital r decreases (assuming that it is equal to the marginal productivity of capital) when the capital/income ratio β increases."

In short, a perpetual drop in the real interest rate r ($\hat{r} < 0$) is a possibility under Piketty's double scenario that $r > g$ and the capital-output ratio β stays put or rises. (The fall in the real interest rate r would get slower and slower, however, so that $r > g$ remains true.)[5] But then it is just as plausible that the income share of capital α actually *drops*.

In summary, at long last: even if Piketty's prediction is right about both $r > g$ and a rising capital-output ratio β, nothing prevents inequality from getting better.

What level of inequality is fair when growth is doomed? Piketty also entertains two dark scenarios for growth. For the first doom scenario, suppose output growth comes to a complete standstill with $g = 0$ while Piketty's first tenet turns into $r > g = 0$. Then, by the growth identity in equation (3), $\hat{A} + r\beta\,\hat{k} + \hat{L} = 0$. Solving out for the interest rate implies:

$$r = -\frac{\hat{A} + \hat{L}}{\beta\hat{k}} > 0.$$

For most of the 21st century, demographers project that world population growth will slow down but remain positive, so $\hat{L}$ may become small but not negative. For the argument's sake, set worker population growth to zero, $\hat{L} = 0$. Unless we forget how we used to make things or actively reduce our productivity over time, productivity change $\hat{A}$ may become arbitrarily small but not plausibly negative. Then Piketty's growth doom with $g = 0$ immediately implies by the above

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[5]For the hardcore economists, in this Piketty economy, the so-called Inada condition cannot hold but instead $Af'(k) > g$ for any level of k.

equation that $\hat{k} < 0$: we must let capital decay so that the capital-labor ratio falls. As capital becomes scarcer and scarcer, the real rate of return r on capital will increase because only the most profitable investments with large returns will be realized.

What does this first doom scenario mean for inequality and the change in the income share of capital? As we saw above, $\hat{\alpha} = \hat{r} + \hat{\beta} = (\hat{r} - g) + \hat{k} + \hat{L}$, which simplifies to

$$\hat{\alpha} = \hat{r} + \hat{k}$$

when $g = \hat{L} = 0$. Inequality may worsen, and α rise, if capital becomes extremely scarce because the return on capital r may increase faster than the rate of capital decay. Is that fair? Perhaps not. Instead of recommending redistribution, however, an arguably more imminent policy question is to ask what we can do to raise productivity change $\hat{A}$, and maybe population growth $\hat{L}$, too.

For Piketty's other doom scenario, suppose productivity change and population growth all come to a complete standstill with $\hat{A} = \hat{L} = 0$ but output keeps growing at a rate g, while Piketty's first tenet remains $r > g > 0$. Then, by the growth identity in equation (3), the only source of growth is capital deepening: $g = r\beta\hat{k} > 0$. What does his second doom scenario mean for inequality and the change in the income share of capital? As we saw above, $\hat{\alpha} = \hat{r} + \hat{\beta} = (\hat{r} - g) + \hat{k} + \hat{L}$, which now simplifies to

$$\hat{\alpha} = \hat{r} + \hat{k} - g = \hat{r} + (1 - r\beta)\hat{k}.$$

If capital deepens relatively fast ($\hat{k}$ positive and large) compared to the likely fall in the interest rate ($\hat{r} < 0$), then inequality worsens. However, in this doom scenario the *one and only* source of growth is capital accumulation. Therefore, in this doom's scenario, it is perhaps hard to make the moral argument that capital owners should not receive a relatively large share of income; their deepening capital provides the only source of growth after all.

An afterthought beyond Piketty: A worst case inequality scenario when all sources of growth matter. One plausible possibility is that the 21st century will not be that different from the 20th in that all three sources of growth keep contributing, that is $\hat{A}$, $\hat{k}$ and $\hat{L}$ all matter and growth will remain clearly positive $g > 0$. In other words, suppose Piketty is wrong in his forecasts about $\hat{A}$ and g, and that both rates of change remain strongly in the positive. But do suppose that Piketty is right in his first tenet that $r > g$ will last. Then it follows from equation (3) directly that

$$r > g = \hat{A} + r\beta\hat{k} + \hat{L}.$$

Solving out for r (subtracting $r\beta\hat{k}$ from both sides) and dividing through by $1 - \beta\hat{k}$, we must have

$$r > \frac{\hat{A} + \hat{L}}{1 - \beta\hat{k}}.$$

This logical necessity has an important immediate implication: If neither productivity nor population growth come to a complete halt, and if Piketty is right about $r > g$, then $1 - \beta\hat{k}$ must be positive because we are not allowed to divide by zero or by a negative number in the ratio on the right-hand side of the inequality above.[6] Therefore capital deepening $\hat{k}$ must be bounded above

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[6]Division by zero is not permissible. Suppose $\hat{k} > 1/\beta > 0$ so that the denominator $1 - \beta\hat{k}$ is strictly negative and the inequality in fact becomes

$$r < \frac{\hat{A} + \hat{L}}{1 - \beta\hat{k}} < 0.$$

by $\hat{k} = 1/\beta$. Piketty's long-term evidence, by which a "natural" β is roughly six or seven, capital deepening can be no faster than 16 percent. That would still a very fast rate: capital owners accumulate more assets at a rate 16 percent faster than population growth—a worst case scenario for inequality as it turns out. But the good news is that in this scenario beyond Piketty's doom the economies will actually keep growing and therefore keep lifting people out of poverty, just that unfortunately the capitalists' incomes will get lifted up even faster.

The top 1 percent and the bottom 99 percent. There are more capital owners than just one percent of the overall population. In fact, for a capital-output ratio of β between 6 and 7, and rising, and an interest rate of 6 to 7 percent or so, the capital share in national income α is roughly equal to between 36 and 49 percent. In recent years, the 1 percent top earners in the United States take home close to one-quarter of national income. To fill the gap between that one-quarter and the 36 to 49 percent, there must therefore be more capitalists than just the top 1 percent of income earners. In fact, most workers are also capital owners in their retirement accounts and their home ownership.

The top 1 percent today get close to one-quarter of national income. But that is not even the full story. Much of the recent increase in income inequality in the United States was driven by the performance of the super-top 1 percent within the top 1 percent. If they all faced the same interest rate r as Piketty's main analysis posits, then the division of society into capital owners and labor does not necessarily offer much empirical oomph unless, for some reason outside the two-group theory, asset ownership among the capitalists is extremely diverse. That is in fact the case. But the asset holdings of the super-top 1 percent within the top 1 percent can then not have come about by a simple rule of capital accumulation over generations, which is common to all capital owners in Piketty's world. Can the marked diversity within the top-1 percent plausibly be due to a higher savings rates in the super-top dynasties? Not likely either, for savings rates do not seem to change all that much with income over time and across income groups.

My hunch is that human capital, entrepreneurial ability, some mere luck, and perhaps a good network with privileged access to resources or insider knowledge, may matter quite strongly for recent changes in income inequality. My prediction is that those determinants of incomes will continue to matter for the evolution of earnings diversity in the 21st century. It will matter for policy to what degree income inequality depends on human capital and merit, and to what degree inequality depends on unfairly privileged access to insider jobs or access to insider resources. Those determinants of income inequality will also determine how acceptable income inequality is for our social consensus. Whatever that emerging consensus will be, the division of society into capital owners and the rest appears quite 20th century.

For (however small but) positive $\hat{A}$ and $\hat{L}$, this implies that $r < 0$. That is a contradiction to $r > g > 0$, so $\hat{k} > 1/\beta$ is not a logical possibility.

References

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