Efficient Mechanisms for Level-*k* Bilateral Trading

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Abstract: I revisit Myerson and Satterthwaite's (1983; "MS") classic analysis of mechanism design for bilateral trading, focusing on direct mechanisms and replacing equilibrium with a structural nonequilibrium "level-*k*" model that predicts initial responses to games. Because the choice of mechanism influences the correctness of level-*k* beliefs, requiring level-*k*-incentive-compatibility may involve loss of generality. If only level-*k*-incentive-compatible mechanisms are feasible, MS's characterization of incentive-efficient mechanisms extends qualitatively to level-*k* traders, with one novel feature, "tacit exploitation of predictably incorrect beliefs." If non-level-*k*-incentive-compatible mechanisms are feasible but people best respond to level-*k* beliefs, level-*k*-incentive-efficient mechanisms differ in form and detail from equilibrium-incentive-efficient mechanisms.

Keywords: mechanism design, incentive-efficient bilateral trading, revelation principle, behavioral game theory, level-*k* thinking

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1. INTRODUCTION

This paper revisits Myerson and Satterthwaite's (1983; "MS") classic analysis of mechanism design for bilateral trading with independent private values. I focus on direct mechanisms and replace MS's assumption that traders will play the desired equilibrium in any game the choice of mechanism creates with the assumption that traders will follow a structural nonequilibrium model based on level-*k* thinking. Otherwise I maintain standard behavioral assumptions.

Why study nonequilibrium design? Equilibrium analyses of design have enjoyed tremendous success; and both theory and experiments support the assumption that players in a game who have enough experience with analogous games will have learned to play an equilibrium. Design, however, creates new games, which may lack the clear precedents required for learning; yet a design may still need to work the first time. Even in settings where learning is possible, design may create games too complex for convergence to equilibrium to be behaviorally plausible.

In theory, equilibrium assumptions can still be justified via epistemic arguments (Aumann and Brandenburger 1995). But in experiments that study initial responses to games, subjects' thinking seldom follows the fixed-point or iterated-dominance reasoning that equilibrium usually requires.[2] Instead their thinking favors level-*k* decision rules, which anchor beliefs in a naive model of others' initial responses, called *L0*, and then adjust them via a small number, *k*, of iterated best responses: *L1* best responds to *L0*, *L2* to *L1*, etc. The frequency of *L0* is usually estimated to be small and subjects' levels to be heterogeneous, concentrated on *L1*, *L2*, and *L3*.

Lk for $k > 0$ is rational, with an accurate model of the game. It departs from equilibrium only in basing its beliefs on an oversimplified model of others' decisions. *Lk*'s decisions also respect *k*-rationalizability (Bernheim 1984), so that a level-*k* model can be viewed as a heterogeneity-tolerant refinement of it.[3] As a result, *Lk* mimics equilibrium decisions in two-person games that are dominance-solvable in *k* rounds, but may deviate systematically in other games. Importantly, the fact that a level-*k* model is structural allows it to predict not only that deviations from equilibrium will occur, but also which kinds of game evoke them and what forms they will take.

[2] Crawford, Costa-Gomes, and Iriberri (2013) survey the experimental literature on strategic thinking. Maskin (2011) argues that "the theoretical and practical drawbacks of Nash equilibrium as a solution concept are far less troublesome in problems of mechanism design", because the game can often be chosen to ensure that equilibrium is unique, or even that it is dominance-solvable. But the experiments suggest that neither feature assures equilibrium initial responses in general, and this extends to many games used in theoretical analyses of implementation (Katok, Sefton, and Yavas 2002; Chen and Ledyard 2008). Some researchers argue that using an incentive-compatible mechanism and announcing that truth-telling is an equilibrium avoids the complexity of equilibrium thinking, but people may still wish to check such claims using their own ways of strategic thinking.

[3] In Camerer, Ho, and Chong's (2004) closely related "cognitive hierarchy" model, *Lk* best responds to a mixture of all lower levels. As a result, it may not always respect *k*-rationalizability when $k > 1$.

A level-*k* analysis could add to the usefulness of equilibrium design theory in several ways. A level-*k* model replaces *k*-rationalizability's set-valued predictions with a selection specific enough to permit an analysis directly comparable to an equilibrium analysis, clarifying the role of the equilibrium assumption.[4] A level-*k* analysis could identify settings in which equilibrium conclusions are robust to likely deviations from equilibrium. It could identify other settings where mechanisms that are optimal, assuming equilibrium, are too fragile to perform as well in practice as equilibrium-suboptimal but less fragile mechanisms: an evidence-based way to assess robustness of mechanisms. Finally, a level-*k* analysis might reduce the unrealistic sensitivity of theoretically optimal mechanisms' to distributional and knowledge assumptions (Wilson 1987).

Section 2 reviews the positive starting point for MS's analysis, Chatterjee and Samuelson's (1983; "CS") classic Bayesian equilibrium analysis of bilateral trading with independent private values via double auction. CS characterized the equilibria of the double auction for well-behaved value densities with overlapping supports. In the case of uniform densities, they identified an equilibrium in which traders' bids are linear in their values, and traders shade their bids so trade occurs only if the buyer's value is sufficiently larger than the seller's. In this and other equilibria, with positive probability some beneficial trades do not occur and trading is inefficient ex post.

Section 3 reviews MS's analysis of design. They asked whether the ex post inefficiency CS noted is a flaw of the double auction or a general property of any feasible mechanism that creates the incentives traders need to reveal the private information on which efficient trading depends. Assuming Bayesian equilibrium, MS argued, via the revelation principle (pp. 267-268), that any equilibrium of any feasible mechanism can be viewed as the truthful equilibrium of a direct-revelation mechanism with the same outcomes, so there is no loss of generality in restricting attention to direct mechanisms that are incentive-compatible in the sense that truthful reporting is an equilibrium. For well-behaved value densities, MS then characterized incentive-efficient trading mechanisms, showing that the ex post inefficiency of the double auction cannot be avoided in equilibrium by any feasible mechanism. They also showed that, with uniform value densities and symmetric surplus-sharing in the auction, CS's linear equilibrium, or equivalently the incentive-compatible direct mechanism that mimics its outcomes, is incentive-efficient.

[4] Until recently the alternatives to equilibrium were limited to adding noise to equilibrium predictions; quantal response equilibrium ("QRE"); or rationalizability or *k*-rationalizability. To my knowledge, equilibrium plus noise has seldom been applied to design; see however de Clippel, Saran, and Serrano (2015). QRE has not been applied to design, perhaps because it must be solved for numerically and its predictions are driven by its error structure. With some exceptions, noted below, rationalizability and *k*-rationalizability have not been applied to design, because set-valued predictions impede analysis.

Section 4 defines a level-*k* model for incomplete-information games. I focus on direct mechanisms, those in which players' decisions are conformable to estimates of their values, for two reasons, one practical and one analytical. The simplicity of direct mechanisms makes them especially well suited to applications. And in more exotic games, level-*k* models are unlikely to describe people's thinking (Crawford, Kugler, Neeman, and Pauzner 2009); but evidence to guide a specification of a reliable model for more general games is lacking. With complete information, *L0*'s decisions are usually taken to be uniform random over the feasible decisions. With incomplete information, I take *L0*'s decisions to be uniform over the feasible decisions and *independent of its private value*. As usual I define *L1*, *L2*, etc. via iterated best responses. This extended model gives a reliable account of the behavioral issues nonequilibrium mechanism design must address: how people's thinking deviates from equilibrium, and their "informational naiveté", their imperfect attention to how others' decisions depend on their private information.

Section 5 revisits CS's equilibrium analysis of the double auction using Section 4's level-*k* model, restricting attention to *L1*s or *L2*s, which are empirically the most frequent and illustrate my main points. In the double auction *L1*s' uniform beliefs over the entire value range make them too optimistic about their partners' bids or asks, relative to equilibrium. That makes *L1*s bid or ask too aggressively, which in the double auction reduces efficiency, driving expected total surplus if both traders are *L1*s well below its level in equilibrium. By contrast, *L2*s' beliefs tend to be too pessimistic. That makes *L2*s tend to bid or ask too *un*aggressively, which increases efficiency, raising surplus if both traders are *L2*s well above its level in equilibrium.

These results raise new questions regarding design: For instance, could a designer who knows that all traders are *L1*s improve upon the double auction by designing a mechanism that curtails *L1*s' aggressiveness in it? And could one who knows that all traders are *L2*s improve upon the double auction by further heightening *L2*s' *un*aggressiveness? Section 6 takes up such questions, replacing equilibrium with a level-*k* model. I assume the population level frequencies are known to the designer and, for most of the analysis, concentrated on one level, *L1* or *L2*.

I define incentive-efficiency notions for correct beliefs, but derive incentive constraints from level-*k* beliefs. "Level-k-incentive-compatibility" and "level-k-interim-individual-rationality" are analogous to the standard notions, which I call "equilibrium-incentive-compatibility" and "equilibrium-interim-individual-rationality". I use "incentive-compatible" in the narrow sense, for direct mechanisms in which it is optimal for people to report truthfully, given their beliefs.

Section 6's first main result, Theorem A, shows that if level-*k*-incentive-compatibility is required, with uniform value densities MS's equilibrium-incentive-efficient mechanism is efficient in the set of level-*k*-incentive-compatible mechanisms (which is then independent of *k*) for any population of level-*k* traders with $k > 0$. In this case MS's closed-form solution for incentive-efficient mechanisms is fully robust to replacing equilibrium with a level-*k* model.

Comparing Theorem A with Section 5's analysis shows that, unlike in MS's equilibrium analysis, the choice between the symmetric double auction and MS's equilibrium-incentive-efficient mechanism is *not* neutral for level-*k* traders: With uniform densities MS's mechanism is efficient for *L1*s in the set of *L1*-incentive-compatible mechanisms only if implemented *not* as a double auction but in its *L1*-incentive-compatible direct form. And by contrast, for *L2*s the *non-L2*-incentive-compatible double auction would, if feasible, improve upon the mechanism that is efficient in the set of *L2*-incentive-compatible mechanisms, violating the revelation principle.

Why do mechanisms that are equivalent in MS's equilibrium analysis yield outcomes that differ, and in opposite directions, for *L1*s and *L2*s? The differences stem from Crawford et al.'s (2009) "level-*k* menu effects", whereby the choice of mechanism influences the correctness of level-*k* beliefs.[5] For *L1*s, such menu effects favor the *L1*-incentive-compatible direct mechanism because it rectifies *L1*s' beliefs and counters their aggressiveness. For *L2*s, the menu effects favor the double auction because it preserves *L2*s' beneficial *un*aggressiveness.

The influence of menu effects means that it matters whether level-*k*-incentive-compatibility is required (in the narrow truthful-revelation sense), as in Theorem A; or can be relaxed to allow direct but non-incentive-compatible mechanisms such as the double auction.[6] Some analysts of design have argued that incentive-compatibility is essential in applications (e.g. Abdulkadiroglu and Sönmez 2003 for school choice; Milgrom, Ausubel, Levin, and Segal 2012 for auctions), though mostly in equilibrium analyses where there is no theoretical gain from relaxing it. Others are willing to consider non-incentive-compatible mechanisms like the Boston Mechanism (Erdil and Ergin 2008; Abdulkadiroglu, Che, and Yasuda 2011) or first-price auctions (Myerson 1981). I take no position on whether incentive-compatibility is essential, which is mostly an empirical question. Instead I require it in most of the analysis, but briefly consider relaxing it at the end.

[5] Such an influence is possible because level-*k* beliefs are anchored on *L0* and not rectified by strategic thinking, as equilibrium beliefs are. Because the influence is on beliefs, not preferences, it is not inconsistent per se with rational individual decisions.

[6] By "relaxing level-*k*-incentive-compatibility" I mean allowing direct mechanisms that create incentives to lie for level-*k* beliefs, not dropping optimality. I call the level-*k* notion of incentive-efficiency "efficiency in the set of level-*k*-incentive-compatible mechanisms" when level-*k*-incentive-compatibility is required, and "level-*k*-incentive-efficiency" when it is relaxed.

Theorems B and C adapt MS's (Theorems 1-2) characterization of equilibrium-incentive-efficient mechanisms for well-behaved value densities to characterize mechanisms that are efficient in the set of level-*k*-incentive-compatible mechanisms for a known population of traders concentrated on one level, either *L1* or *L2*. MS's characterization is in this case fully robust to replacing equilibrium with a level-*k* model, even though their analysis relies on the behaviorally strong assumption that traders play an equilibrium that is a fixed-point in a high-dimensional strategy space. It follows that the design features that favor incentive-efficiency in MS's equilibrium analysis also favor it in the level-*k* analysis, with different weights due to level-*k* beliefs. However, MS's Corollary, that no incentive-compatible, interim individually rational mechanism is ex post efficient with probability one, does not fully extend to the level-*k* analysis.

Theorems B and C also reveal an efficiency-enhancing design feature that can only arise in a structural nonequilibrium analysis, tacit exploitation of predictably incorrect beliefs ("TEPIB"): "predictably incorrect" in that the level-*k* model predicts the distribution of traders' deviations from equilibrium; "exploitation" in the benign sense of using traders' nonequilibrium responses for their own benefit; and "tacit" in that the mechanism does not actively mislead traders.

As in MS's analysis, mechanisms that are efficient in the set of level-*k*-incentive-compatible mechanisms can be solved for in closed form only with uniform value densities, for which they happen to induce level-*k* beliefs that are correct, neutralizing TEPIB. To assess TEPIB's influence and importance, I compute such mechanisms for *L1*s, for representative combinations of linear value densities. Such mechanisms exploit TEPIB but are otherwise similar in most respects to equilibrium-incentive-efficient mechanisms. When buyers' true densities make *L1*s' beliefs pessimistic, such mechanisms tend to implement larger trading regions and more efficient outcomes than equilibrium-incentive-efficient mechanisms. When *L1*s' beliefs are optimistic, such mechanisms tend to implement smaller trading regions.

I next briefly consider relaxing the assumption that the population is concentrated on one level, still assuming the designer knows the population mixture of levels and requiring level-*k*-incentive-compatibility. If the designer is allowed to offer only a single, direct mechanism, rather than a menu, only trivial mechanisms can fully screen traders' levels and values simultaneously. That is normally suboptimal; and there appears to be no simple structure on which levels and values should be screened. If *L1*s (respectively *L2*s) predominate, it is likely that a mechanism that is efficient in the set of level-*k*-incentive-compatible mechanisms for a known population concentrated on *L1* (respectively *L2*), is then optimal for the nearly homogeneous mixture of levels. For less extreme distributions, the picture is unclear.

Finally, I briefly consider relaxing level-*k*-incentive-compatibility, returning to the assumption of a known population of traders concentrated on one level, *L1* or *L2*. As a tractable approximation of what is achievable via any feasible direct mechanism, I study double auctions with reserve prices chosen by the designer and assume uniform value densities. If level-*k* traders anchor their beliefs on the menu of possible bids as restricted by the reserve prices, instead of on the full range of possible values, for *L1*s such an auction can improve upon a mechanism that is efficient in the set of *L1*-incentive-efficient mechanisms, by taking fuller advantage of TEPIB. For *L2*s Section 5's analysis already shows that a double auction can improve upon a mechanism that is efficient in the set of *L2*-incentive-efficient mechanisms without reserve prices; and calculations show that reserve prices allow more improvement. Overall, relaxing level-*k*-incentive-compatibility allows level-*k*-incentive-efficient mechanisms to differ from equilibrium-incentive-efficient mechanisms in form as well as detail, sometimes with large efficiency gains.

In addition to CS's and MS's analyses and Crawford and Iriberri's (2007) level-*k* analysis of sealed-bid auctions, this paper builds on Crawford et al.'s (2009) level-*k* analysis of optimal independent-private-value auctions, which builds in turn on Myerson's (1981) classic equilibrium analysis. Crawford et al. (2009) showed for auctions, as I show here for trading, that level-*k* design involves more than implementing equilibrium outcomes under weaker behavioral assumptions: A second-price auction may seem superior to a first-price auction because it yields the equilibrium outcome for any population of level-*k* bidders. But revenue-equivalence fails for level-*k* bidders, and TEPIB may allow a first-price auction to yield higher expected revenue.

Kneeland (2013), Gorelkina (2015), and de Clippel, Saran, and Serrano (2015) use level-*k* models to study design in settings including bilateral trading. Saran (2011a) studies MS's design problem when some traders report truthfully without regard to incentives. Hagerty and Rogerson (1987) and Bulow and Roberts (1989) study dominant-strategy implementation in MS's trading environment, the latter achieving positive results by giving up ex post budget balance. In more abstract settings, Mookherjee and Reichelstein (1992) study dominant-strategy implementation; and Matsushima (2007, 2008) studies implementation via finitely iterated dominance.

Finally, Glazer and Rubinstein (1998), Neeman (2003), Eliaz and Spiegler (2006, 2007, 2008), and Wolitzky (2014) study design with "behavioral" individual decisions or judgment.

2. EQUILIBRIUM BILATERAL TRADING VIA DOUBLE AUCTION

Following CS and MS, I consider bilateral trading between a potential seller and buyer of an indivisible object, in exchange for an amount of money to be determined. The traders' von Neumann-Morgenstern utility functions are quasilinear in money, so they are risk-neutral and have well-defined money values for the object. Denote the buyer's value *V* and the seller's *C* (for "cost"; but I sometimes use "value" generically for *C* as well as *V*). *V* and *C* are independently distributed, with probability densities *f*(*V*) and *g*(*C*) that are strictly positive on their supports, and probability distribution functions *F*(*V*) and *G*(*C*). CS and MS allowed traders' value distributions to have any bounded overlapping supports, but for simplicity and with no important loss of generality, I take their supports to be identical and normalize them to [0, 1].

CS study a double auction, in which traders make simultaneous money offers. If the buyer's offer *b* (for "bid") exceeds the seller's offer *a* ("ask"), they exchange the object for a price that is a weighted average of *a* and *b*. CS allowed weights from 0 to 1, but as in MS's analysis I focus on the symmetric case with weights ½. Then, if $b \geq a$, the buyer acquires the object at price (*a* + *b*)/2, the seller's utility is (*a* + *b*)/2, and the buyer's is *V* - (*a* + *b*)/2. If $b < a$, the seller retains the object, no money changes hands, the seller's utility is *C*, and the buyer's is 0.

As CS noted, this game has many Bayesian equilibria. I follow them and the subsequent literature in focusing on equilibria in which trade occurs with positive probability, and traders' strategies are bounded above and below, strictly increasing, and (except possibly at the boundaries) differentiable. Denote the buyer's bidding strategy *b*(*V*) and the seller's asking strategy *a*(*C*). An equilibrium buyer's bid $b_*(V)$ must maximize, over *b* ϵ [0, 1]

$$\int_0^b \left(V - \left[\frac{a+b}{2}\right]\right) g(a_*^{-1}(a))da + \int_b^1 0\, da,$$

where $g(a_*^{-1}(a))$ is the density of an equilibrium seller's ask $a_*(C)$ induced by the seller's value density *g*(*C*). Similarly, an equilibrium seller's ask $a_*(C)$ must maximize, over *a* ϵ [0, 1]

$$\int_a^1 \left[\frac{a+b}{2}\right] f(b_*^{-1}(b))db + \int_0^a C\, f(b_*^{-1}(b))db,$$

where $f(b_*^{-1}(b))$ is the density of an equilibrium buyer's bid $b_*(V)$ given the value density *f*(*V*).

In the leading case where traders' value densities *f*(*V*) and *g*(*C*) are uniform, CS gave a closed-form solution for a linear equilibrium, which was also important in MS's analysis. Given my normalization of the supports of *f*(*V*) and *g*(*C*) to [0, 1], in this equilibrium $b_*(V) = 2V/3 +$

$1/12$ unless $V < 1/4$, in which case $b_*(V)$ can be anything that does not lead to trade; and $a_*(C) = 2C/3 + 1/4$ unless $C > 3/4$, when $a_*(C)$ can be anything that does not lead to trade.

With those strategies, trade takes place if and only if $2V/3 + 1/12 \geq 2C/3 + 1/4$, or $V \geq C + 1/4$, at price $(V + C)/3 + 1/6$. Thus with positive probability the outcome is ex post inefficient. Even so, MS showed that in this case the double auction implements an ex ante incentive-efficient outcome, by showing that, assuming Bayesian equilibrium, no mechanism can ex ante Pareto-dominate the linear equilibrium of the double auction (MS's p. 277 or Section 3.3 below). The linear equilibrium yields ex ante probability of trade $9/32 \approx 28\%$ and expected surplus $9/64 \approx 0.14$, less than the maximum interim (after traders observe own values, before they observe the outcome) individually rational probability of trade of 50% and expected total surplus $1/6 \approx 0.17$.

3. EQUILIBRIUM MECHANISM DESIGN FOR BILATERAL TRADING

Assuming Bayesian equilibrium, MS characterized ex ante incentive-efficient mechanisms in CS's trading environment, allowing any feasible mechanism and taking into account the need to ensure interim individual rationality. I now review MS's analysis, using my notation.

3.1 The revelation principle

In a *direct* mechanism traders make simultaneous reports of their values, which I denote v and c to distinguish them from traders' true values V and C, and those reports then determine the outcome. MS's assumption that traders will play the desired equilibrium in any game the designer's choice of mechanism creates allows an important simplification of their analysis via the revelation principle. Because that simplification must be reconsidered in the level-k analysis, I quote MS's (pp. 267-268) equilibrium argument for the revelation principle here:

> We can, without any loss of generality, restrict our attention to incentive-compatible direct mechanisms. This is because, for any Bayesian equilibrium of any bargaining game, there is an equivalent incentive-compatible direct mechanism that always yields the same outcomes (when the individuals play the honest equilibrium)….[w]e can construct [such a] mechanism by first asking the buyer and seller each to confidentially report his valuation, then computing what each would have done in the given equilibrium strategies with these valuations, and then implementing the outcome (transfer of money and object) as in the given game for this computed behavior. If either individual had any incentive to lie to us in this direct mechanism, then he would have had an incentive to lie to himself in the original game, which is a contradiction of the premise that he was in equilibrium in the original game.

3.2 Equilibrium-incentive-compatible direct trading mechanisms

When traders are risk-neutral, the payoff-relevant outcomes of a direct mechanism are completely described by two outcome functions, $p(\cdot,\cdot)$ and $x(\cdot,\cdot)$, where if the buyer and seller report values v and c, then $p(v, c)$ is the probability the object is transferred from seller to buyer and $x(v, c)$ is the expected monetary payment from buyer to seller. For a direct mechanism $p(\cdot,\cdot)$ and $x(\cdot,\cdot)$, define the buyer's and seller's expected monetary payments, probabilities of trade, and utilities as functions of their value reports v and c (hats denote variables of integration):

$$X_B(v) = \int_0^1 x(v,\hat{c})g(\hat{c})d\hat{c}, \qquad X_S(c) = \int_0^1 x(\hat{v},c)f(\hat{v})d\hat{v},$$

(3.1) $$P_B(v) = \int_0^1 p(v,\hat{c})g(\hat{c})d\hat{c}, \qquad P_S(c) = \int_0^1 p(\hat{v},c)f(\hat{v})d\hat{v},$$

$$U_B(v) = vP_B(v) - X_B(v), \qquad U_S(c) = X_S(c) - cP_S(c).$$

Although the outcome functions take only on traders' reported values as arguments, traders' expected utilities also depend on their true values. Thus the mechanism $p(\cdot,\cdot)$, $x(\cdot,\cdot)$ (with the qualification "direct" omitted from now on) is *incentive-compatible* if and only if truthful reporting is an equilibrium; that is, if for every V, v, C, and c in [0, 1],

(3.2) $$U_B(V) \geq VP_B(v) - X_B(v) \text{ and } U_S(C) \geq X_S(c) - CP_S(c).$$

Similarly, $p(\cdot,\cdot)$, $x(\cdot,\cdot)$ is *interim individually rational* if and only if for every V and C in [0, 1],

(3.3) $$U_B(V) \geq 0 \text{ and } U_S(C) \geq 0.$$

MS's Theorem 1. *For any incentive-compatible mechanism,*

(3.4) $$U_B(0) + U_S(1) = \min_{V\in[0,1]} U_B(V) + \min_{C\in[0,1]} U_S(C)$$

$$= \int_0^1\int_0^1 \left(\left[V - \frac{1-F(V)}{f(V)}\right] - \left[C + \frac{G(C)}{g(C)}\right]\right) p(V,C)f(V)g(C)dCdV.$$

Furthermore, if $p(\cdot,\cdot)$ is any function mapping [0, 1]×[0, 1] into [0, 1], then there exists a function $x(\cdot,\cdot)$ such that (p, x) is incentive-compatible and interim individually rational if and only if $P_B(\cdot)$ is weakly increasing, $P_S(\cdot)$ is weakly decreasing, and

(3.5) $$0 \leq \int_0^1\int_0^1 \left(\left[V - \frac{1-F(V)}{f(V)}\right] - \left[C + \frac{G(C)}{g(C)}\right]\right) p(V,C)f(V)g(C)dCdV.$$

Proof. MS (pp. 269-270) showed that (3.2) implies that $P_B(\cdot)$ is weakly increasing and $P_S(\cdot)$ is weakly decreasing, and yields necessary and sufficient conditions for incentive-compatibility:

(3.6) $$U_B(V) = U_B(0) + \int_0^V P_B(v)dv \quad \text{and} \quad U_S(C) = U_S(1) + \int_C^1 P_S(c)dc.$$

(3.6) implies that $U_B(V)$ is weakly increasing and $U_S(C)$ is weakly decreasing, so $U_B(0) \geq 0$ and $U_S(1) \geq 0$ suffice for interim individual rationality as in (3.3). MS (p. 270) next showed that

(3.7) $$\int_0^1 \int_0^1 (V - C)\, p(V,C) f(V) g(C) dC dV$$

$$= U_B(0) + U_S(1) + \int_0^1 \int_0^1 [\big(1 - F(V)\big)g(C) + G(C)f(V)]\, p(V,C) dC dV.$$

(3.7) implies (3.4) and, given (3.3), (3.5). Finally, given (3.5) and that $P_B(\cdot)$ is weakly increasing and $P_S(\cdot)$ is weakly decreasing, it is always possible, following MS (pp. 270-271), to construct a transfer function $x(\cdot, \cdot)$ such that (p, x) is an incentive-compatible and interim individually rational mechanism. Q.E.D.

3.3 Equilibrium-incentive-efficient trading mechanisms

Given that ex post efficiency cannot be guaranteed for an incentive-compatible, interim individually rational mechanism, it is natural to consider the extent to which incentive-compatibility limits efficiency. MS's Theorem 2 addresses this question. To state it, they define

(3.8) $$\Phi(V,\alpha) = V - \alpha \frac{1-F(V)}{f(V)} \quad \text{and} \quad \Gamma(C,\alpha) = C + \alpha \frac{G(C)}{g(C)},$$

$$p^\alpha(V,C) = 1 \quad \text{if} \quad \Gamma(C,\alpha) \leq \Phi(V,\alpha), \text{ and } p^\alpha(V,C) = 0 \quad \text{if} \quad \Gamma(C,\alpha) > \Phi(V,\alpha).$$

MS's Theorem 2. *If there exists an incentive-compatible mechanism (p, x) such that* $U_S(1) = U_B(0) = 0$ *and* $p = p^\alpha(V,C)$ *for some* $\alpha \in [0,1]$, *then that mechanism maximizes the expected gains from trade among all incentive-compatible, interim individually rational mechanisms. Furthermore, if* $\Phi(V,1)$ *and* $\Gamma(C,1)$ *are increasing on* [0, 1], *then such a mechanism must exist.*

Proof. Note that $p^0(V,C)$ would yield an ex post efficient allocation, and $p^1(V,C)$ would maximize the slack in (3.5), which functions as a kind of "incentive budget constraint". $p^1(V,C)$ wastes surplus, and MS's Corollary 1 (restated below) shows that $p^0(V,C)$ is unaffordable. The goal is an optimal compromise between those extremes, choosing the (V, C) combinations on which to trade that yield the largest expected gains per unit of incentive cost. Thus, consider the problem of choosing the function $p(\cdot, \cdot)$ to maximize the expected gains from trade

$$\int_0^1 \int_0^1 (V - C)\, p(V,C) f(V) g(C) dC dV$$

subject to (3.5). If the solution to this problem happens to yield functions $P_B(V)$ and $P_S(C)$ that are monotone increasing and decreasing, respectively, then by MS's Theorem 1, the solution

$p(\cdot, \cdot)$ is associated with a mechanism that maximizes the expected gains from trade among all incentive-compatible, interim individually rational mechanisms. Optimality plainly requires $U_S(1) = U_B(0) = 0$, and that (3.5) holds with equality at the solution. Further, if $\Phi(V, 1)$ and $\Gamma(C, 1)$ are increasing in V and C respectively, then $\Phi(V, \alpha)$ and $\Gamma(C, \alpha)$ are similarly increasing for all $\alpha \in [0,1]$. Thus $p^\alpha(V, C)$, which is defined so that varying α selects the trades that make the greatest contribution to expected gains from trade relative to their unit incentive cost in (3.5), is increasing in V and decreasing in C, and the associated $P_B(V)$ and $P_S(C)$ functions have the required monotonicity properties. Finally, MS (p. 276) show that there always exists an α such that (3.5) holds with equality and $p^\alpha(V, C)$ yields an incentive-compatible mechanism. Q.E.D.

MS's Corollary 1. *If traders have positive value densities with overlapping supports, no incentive-compatible, interim individually rational mechanism can be ex post efficient with probability one.*

Proof. Computations that are a special case of those in Section 6.3's level-*k* analysis of this issue show that the conditions for ex post efficiency with probability one violate (3.5). Q.E.D.

With uniform value densities, MS's Theorem 2 allows a closed-form solution for the incentive-compatible, individually rational mechanism that maximizes expected total surplus. With uniform densities, (3.8)'s criterion for $p^\alpha(V, C) = 1,\ \Gamma(C, \alpha) \leq \Phi(V, \alpha)$, reduces to

(3.9) $$v - c \geq \frac{\alpha}{1+\alpha}$$

and (3.4) with equality reduces to

(3.10) $$0 = \int_{\frac{\alpha}{1+\alpha}}^{1} \int_0^{V-\frac{\alpha}{1+\alpha}} (2V - 1 - 2C)\, dCdV = \frac{3\alpha - 1}{6(1+\alpha)^3},$$

which implies that $\alpha = 1/3$ (MS, p. 277).The incentive-efficient direct mechanism then transfers the object if and only if traders' reported values satisfy $v \geq c + 1/4$, at price $(v + c)/3 + 1/6$. With truthful reporting, this outcome function is identical to that of CS's linear double-auction equilibrium: Although the double auction is not incentive-compatible, traders shade their bids in equilibrium to mimic the outcomes of MS's incentive-efficient direct mechanism.[7] The resulting ex ante probability of trade is 9/32 ≈ 28% and the expected total surplus is 9/64 ≈ 0.14.

[7] However, Satterthwaite and Williams (1989, Theorem 5.1) showed that for generic value densities CS's double auction equilibria are incentive-*in*efficient. Thus MS's remarkable result for the case of uniform value densities is a coincidence.

4. A LEVEL-*K* MODEL FOR INCOMPLETE-INFORMATION GAMES

This section specifies a level-*k* model for CS's and MS's trading environment. I focus on direct mechanisms, in which players' decisions are conformable to value estimates, for two reasons. The simplicity of direct mechanisms makes them especially well suited to applications. And in more exotic games, level-*k* models are unlikely to describe people's thinking (Crawford et al. 2009); but evidence to guide a specification of a model for general games is lacking.

Recall that a level-*k* player anchors its beliefs in an *L0* that represents a naive model of other players' responses, with which it assesses the payoff implications of its own decisions before thinking about others' incentives (Crawford et al. 2013, Sections 2.4 and 3). *Lk* then adjusts its beliefs via iterated best responses: *L1* best responds to *L0*, *L2* to *L1*, and so on.

In complete-information games *L0* is usually taken to be uniformly randomly distributed over the range of feasible decisions. With incomplete information, following Camerer et al. (2004), Crawford and Iriberri (2007), and Crawford et al. (2009), I take *L0*'s decisions as uniform over the feasible decisions and *independent of its own value*. As usual I define *L1*, *L2*, etc. via iterated best responses. I also assume that a player's level is independent of its value, a plausible provisional assumption. The resulting level-*k* model dates from Milgrom and Stokey's (1982) notions of "Naïve Behavior" (equivalent to an *L1* best responding to such an *L0*) and "First-Order Sophistication" (*L2*), which they speculated might explain zero-sum trades despite their equilibrium no-trade/"Groucho Marx" theorem.[8]

There is a growing body of evidence that this extended level-*k* model gives a reliable account of the main patterns of people's non-equilibrium thinking and informational naiveté, or imperfect attention to how others' decisions depend on their private information, as often observed in experiments and in the field. For instance, Crawford and Iriberri (2007) showed that the model gives a coherent econometric account of subjects' overbidding and vulnerability to the winner's curse in their initial responses in the classic auction experiments.[9] Camerer et al. (2004)

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[8] It is easy to imagine alternative specifications. For instance, an *L0* buyer's bid or seller's ask might be assumed to be uniformly distributed below (above) its value, eliminating weakly dominated bids. But *L0* represents not a real player but his naïve model of others, whose values he has not observed. Such a model could involve reasoning contingent on others' values, but the experiments mentioned next strongly suggest that most people are averse to such contingent reasoning. Another alternative would be to assume that *L0* optimizes given its true value, which is a well-defined notion for direct-revelation games. Such a "truthful" *L0* would make level-*k*-incentive-compatibility identical to equilibrium-incentive-compatibility. My analysis goes through unchanged if the value densities have overlapping but not identical supports, if *L0* anchors on the overlapping part.

[9] In a detailed econometric horse race, Crawford and Iriberri found that with minor exceptions, this level-*k* model fits better than equilibrium plus noise; Eyster and Rabin's (2005) "cursed equilibrium"; QRE; or Kagel and Levin's (1986) "naïve" bidders. Crawford and Iriberri (2007) also allowed *Lk*s that best respond to a truthful *L0*, but they fit most subjects' behavior poorly.

suggested that a cognitive hierarchy analogue of this model could explain zero-sum betting, and Brocas et al. (2014) reported experimental evidence in which the level-*k* version of the model explains both the patterns by which subjects' betting deviates from equilibrium and their searches for hidden but freely accessible payoff information (which the level-*k* model predicts). Finally, Brown, Camerer, and Lovallo (2012) use this model to explain film-goers' failure to draw negative inferences from studios' withholding weak movies from critics before release.

5. LEVEL-*K* BILATERAL TRADING VIA DOUBLE AUCTION

This section considers bilateral trading via the double auction using the level-*k* model, restricting attention for simplicity to homogeneous populations of *L1*s or *L2*s, which are empirically the most frequent and illustrate my main points. For *L1*s the analysis applies to general value densities, except that I assume that a trader's level is independent of his value. For *L2*s I focus on the leading case of uniform value densities, which illustrates my main points. Denote the buyer's bidding strategy $b_i(V)$ and the seller's asking strategy $a_i(C)$, for levels $i = 1, 2$.

5.1 L1 *traders*

An *L1* buyer believes that the seller's *L0* ask is uniformly distributed on [0, 1]. Thus an *L1* buyer's bid $b_1(V)$ must maximize, over $b \in [0, 1]$

$$\int_0^b \left[V - \frac{a+b}{2}\right] da + \int_b^1 0\, da.$$

The optimal *L1* strategies are increasing, so the event $a = b$ can again be ignored; and the second-order condition for *L1*'s problem is always satisfied. Solving the first-order condition yields, for any value densities, $b_1(V) = 2V/3$, with range [0, 2/3]. Thus, boundaries aside, an *L1* buyer bids 1/12 more aggressively (bids less) than an equilibrium buyer with uniform value densities: An *L1* buyer's naïve model of the seller systematically underestimates the distribution of the seller's upward-shaded ask, relative to equilibrium, inducing the buyer to underbid.[10]

Similarly, an *L1* seller's ask $a_1(C)$ must maximize, over $a \in [0, 1]$

$$\int_a^1 \frac{a+b}{2} db + \int_0^a C\, db.$$

[10] Here and in Section 5.2, compare Crawford and Iriberri's (2007) analysis of *L1* and *L2* bidding in first-price auctions. These conclusions can be reversed if the seller's value density slopes downward sufficiently steeply. Despite the multiplicity of equilibria in the double auction, the level-*k* model makes generically unique predictions, conditional on the frequencies of traders' rules. An *L1* buyer's or seller's optimal bidding strategy is independent of the value densities—unlike an *L2*'s, which depends on its partner's density, or an equilibrium trader's, which depends on both traders' densities. Despite the multiplicity of equilibria in the double auction, the level-*k* model makes generically unique predictions conditional on the level frequencies.

The first-order condition yields, for any densities, $a_1(C) = 2C/3 + 1/3$, with range [1/3, 1]. An *L1* seller asks 1/12 more aggressively (asks more) than an equilibrium seller with uniform densities.

To sum up, with uniform value densities, *L1* traders' bidding strategies have the same slopes as equilibrium traders' strategies, but are 1/12 more aggressive. When an *L1* buyer meets an *L1* seller, trade takes place if and only if $V \geq C + 1/2$, so the value gap needed for trade is 1/4 larger than for equilibrium traders, and ex post efficiency is lost for more value combinations. An *L1* buyer's and seller's ex ante probability of trade is 1/8 = 12.5%, less than the equilibrium probability 9/32 ≈ 28% and the maximum (interim) individually rational probability 50%.

5.2 L2 *traders*

An *L2* buyer's bid $b_2(V)$ must maximize, over $b \in [0, 1]$

$$\int_0^b \left[V - \frac{a+b}{2}\right] g(a_1^{-1}(a))\, da + \int_b^1 0\, da,$$

where $g(a_1^{-1}(a))$ is the density of an *L1* seller's ask $a_1(C)$ induced by the value density $g(C)$.

If, for instance, $g(C)$ is uniform, an *L2* buyer believes that the seller's ask $a_1(C) = 2C/3 + 1/3$ is uniformly distributed on [1/3, 1], with density 3/2 there and 0 elsewhere. It thus believes that trade requires $b > 1/3$. For $V \leq 1/3$ it is therefore optimal to bid anything it thinks yields 0 probability of trade. In the absence of dominance among such strategies, I set $b_2(V) = V$ for $V \in [0, 1/3]$. For $V > 1/3$, if $g(C)$ is uniform, an *L2* buyer's bid $b_2(V)$ must maximize over $b \in [1/3, 1]$

$$\int_{1/3}^{\mathrm{b}} \left[V - \frac{a+b}{2}\right] (3/2)\mathrm{d}a.$$

The optimal *L2* strategies are increasing, so the event $a = b$ can again be ignored. The second-order condition is again satisfied. Solving the first-order condition $(3/2)(V - \mathrm{b}) - (3/4)(V - 1/3) = 0$ yields $b_2(V) = 2V/3 + 1/9$ for $V \in [1/3, 1]$, with range [1/3, 7/9].

Comparing an *L2* buyer's optimal strategy to an equilibrium or *L1* buyer's optimal strategy, boundaries aside, with uniform value densities an *L2* buyer bids 1/36 less aggressively (bids more) than an equilibrium buyer, and 1/9 less aggressively than an *L1* buyer: An *L2* buyer's model of the seller systematically overestimates the distribution of the seller's upward-shaded ask, relative to equilibrium, inducing the buyer to overbid.

An *L2* seller's ask $a_2(C)$ must maximize over $a \in [0, 1]$

$$\int_a^1 \frac{a+b}{2} f\left(b_1^{-1}(b)\right) db + \int_0^a C\, f\left(b_1^{-1}(b)\right) db,$$

where $f(b_1^{-1}(b))$ is the density of an *L1* buyer's bid $b_1(V)$ induced by the value density $f(V)$.

If, for instance, $f(V)$ is uniform, an *L2* seller believes that the buyer's bid $b_1(V) = 2V/3$ is uniform on [0, 2/3], with density 3/2 there and 0 elsewhere. It thus believes trade requires $a <$ 2/3. For $C \geq 2/3$ it is therefore optimal for an *L2* seller to bid anything it thinks yields zero probability of trade. In the absence of dominance among such strategies, I set $a_2(C) = C$ for $C \in$ [2/3, 1]. For $C < 2/3$, an *L2* seller's ask $a_2(C)$ must maximize over $a \in [0, 2/3]$

$$\int_a^{\frac{2}{3}} \frac{a+b}{2}(3/2)db + \int_0^a C\,(3/2)db.$$

The second-order condition is satisfied, and the first-order condition $(3/2)(a\text{-}C) + (3/2)(2/3 - C)/2 = 0$ yields $a_2(C) = 2C/3 + 2/9$ for $C \in [0, 2/3]$, with range [2/9, 2/3].

Comparing an *L2* seller's optimal strategy to an equilibrium or *L1* seller's optimal strategy, boundaries aside, with uniform value densities an *L2* seller asks 1/36 less aggressively (asks less) than an equilibrium seller, and 1/9 less aggressively than an *L1* seller.

To sum up, with uniform value densities *L2* traders' strategies again have the same slope as equilibrium traders' strategies, but are 1/36 less aggressive. When an *L2* buyer meets an *L2* seller, trade takes place if and only if $V \geq C + 1/6$, so the value gap needed for trade is 1/12 less than for equilibrium traders (1/3 less than for *L1*s), and ex post efficiency is lost for fewer values. The ex ante probability of trade is 25/72 ≈ 35%, higher than the equilibrium probability 9/32 ≈ 28% but still well below the maximum (interim) individually rational probability 50%.[11]

6. MECHANISM DESIGN FOR LEVEL-*K* BILATERAL TRADING

Section 5's analysis shows that with uniform value densities, in the double auction *L1* traders are too optimistic about their partners' bids or asks, relative to the correct beliefs of CS's linear equilibrium. That makes *L1*s bid or ask too aggressively, which drives expected total surplus if both traders are *L1*s well below its equilibrium level. By contrast, *L2*s are too pessimistic and bid too *un*aggressively, which raises surplus if both traders are *L2*s well above its equilibrium level.

Could a designer who knows all traders are *L1*s design a mechanism that improves upon the double auction by curtailing their aggressiveness in it? And could one who knows all traders are *L2*s design a mechanism that improves upon the double auction by further heightening their

[11] With uniform value densities, when an *L2* buyer meets an *L1* seller, or vice versa, trade takes place when $V \geq C + 1/3$, so the necessary value gap is 1/12 more than for a pair of equilibrium traders, 1/6 more than for a pair of *L2*s, but 1/6 less than for a pair of *L1*s. Assuming that the distribution of levels is the same for buyers and sellers, the expected frequency of trade is determined by the population's average level because traders' contributions to the value gap are additive in these examples.

*un*aggressiveness? This section takes up such questions, reconsidering MS's design problem, focusing on direct mechanisms and replacing equilibrium with a level-k model. I assume the designer knows the population level frequencies, and for most of the analysis that they are concentrated on either *L1* or *L2*. As in MS's and most analyses of design, I ignore decision noise.

I define incentive-efficiency notions for correct beliefs, but derive incentive constraints from level-k beliefs. I use "incentive-compatible" in the narrow sense, for direct mechanisms in which it is optimal for people to report truthfully, given their beliefs. "Level-k-incentive-compatibility" and "level-k-interim-individual-rationality" are analogous to the standard notions, which I call "equilibrium-incentive-compatibility" and "equilibrium-interim-individual-rationality".

6.1 Mechanisms that are efficient in the set of level-k-incentive-compatible mechanisms with uniform value densities

First suppose that level-k-incentive-compatibility is required, and consider the leading special case of uniform value densities. Theorem A shows that MS's equilibrium-incentive-efficient direct mechanism, which then mimics CS's linear double-auction equilibrium, is efficient in the set of level-k-incentive-compatible mechanisms (which is then independent of k) for any population of level-k traders with $k > 0$. In this case MS's closed-form solution for the incentive-efficient mechanism is fully robust to relaxing equilibrium in favor of a level-k model.[12]

Theorem A. *With uniform value densities, MS's equilibrium-incentive-efficient direct mechanism is efficient in the set of level-k-incentive-compatible mechanisms for any population of levels with k > 0, known or concentrated on one level or not.*

Proof. This result follows from Theorem B below, but I give a more direct proof here. With uniform value densities, in the truthful equilibrium of MS's equilibrium-incentive-efficient direct mechanism, each trader faces a uniform distribution of the other's reports. For a different reason, *L1* traders best respond to *L0*s that also imply uniform distributions. *L1* traders' conditions for individual rationality and incentive-compatibility ((6.2)-6.3) and (6.5)-(6.6) below) then coincide with the analogous conditions for equilibrium traders ((3.2)-(3.3) and (3.5)-(3.6); MS's (2)-(4)). The equilibrium-incentive-efficient mechanism is therefore efficient in the set of *L1*-incentive-compatible mechanisms, both defined for correct beliefs. Further, because the latter mechanism makes *L1*s report truthfully, *L2*s' incentive conditions coincide with *L1*s', as do *Lk*s' ad

[12] Theorem A's conclusion holds trivially for any nonequilibrium model in which all players best respond to correct beliefs.

infinitum. Thus the equilibrium-incentive-efficient mechanism is efficient in the set of level-*k*-incentive-compatible direct mechanisms for any distribution of level-*k* traders with $k > 0$. Q.E.D.

6.2 Level-k menu effects

Comparing Theorem A with Section 5's analysis shows that, unlike in the equilibrium analysis, the choice between the symmetric double auction and MS's equilibrium-incentive-efficient mechanism is *not* neutral for level-*k* traders: With uniform densities MS's mechanism is efficient for *L1*s in the set of *L1*-incentive-compatible mechanisms only if implemented *not* as a double auction but in its *L1*-incentive-compatible direct form. And by contrast, for *L2*s the *non-L2*-incentive-compatible double auction would, if feasible, improve upon the mechanism that is efficient in the set of *L2*-incentive-compatible mechanisms, violating the revelation principle.

Why do mechanisms that are equivalent via the revelation principle in the equilibrium analysis yield outcomes that differ, and in opposite directions, for *L1*s and *L2*s? The differences stem from Crawford et al.'s (2009) level-*k* menu effects, whereby the choice of mechanism influences the correctness of level-*k* beliefs. For *L1*s such menu effects favor the *L1*-incentive-compatible direct mechanism, because it rectifies *L1*s' beliefs and counters their aggressiveness. For *L2*s they favor the double auction, because it preserves *L2*s' beneficial *un*aggressiveness.

With menu effects, it matters whether level-*k*-incentive-compatibility is required (in the narrow truthful-revelation sense; see footnote 6) as in Theorem A, or can be relaxed to allow direct non-incentive-compatible mechanisms such as the double auction. Sections 6.1-6.4 assume that level-*k*-incentive-compatibility is required, and Section 6.5 relaxes it. When incentive-compatibility is required, I call the associated incentive-efficiency notion "efficiency in the set of level-*k*-incentive-compatible mechanisms"; and when not, I call it "level-*k*-incentive-efficiency".

6.3 Mechanisms that are efficient in the set of level-k-incentive-compatible mechanisms with general value densities and known populations concentrated on one level

This section extends Section 6.1's analysis to general well-behaved value densities and known populations of *L1s* or *L2s* concentrated on one level. As in MS's analysis, the payoff-relevant outcomes of a direct mechanism are $p(v, c)$, the probability the object is transferred, and the expected payment $x(v, c)$, where v and c are the buyer's and seller's reported values.

For any mechanism (p, x), let $f^k(v; p, x)$ and $F^k(v; p, x)$ denote the density and distribution function of an *Lk* seller's beliefs, and $g^k(c; p, x)$ and $G^k(c; p, x)$ the density and distribution function of an *Lk* buyer's. With *L0* uniform random on [0, 1], $f^1(v; p, x) \equiv 1$ and $g^1(c; p, x) \equiv$

1. If $\beta_1(V; p, x)$ is an *L1* buyer's response to (*p*, *x*) with value *V* and $\alpha_1(C; p, x)$ is an *L1* seller's response to (*p*, *x*) with cost *C*, $f^2(v; p, x) \equiv f(\beta_1^{-1}(v; p, x))$ and $g^2(c; p, x) \equiv g(\alpha_1^{-1}(c; p, x))$. I sometimes suppress the dependence of $f^2(v; p, x)$ and $g^2(c; p, x)$ on (*p*, *x*) when it is fixed.

Write the buyer's and seller's expected monetary payments, probabilities of trade, and utilities as functions of their value reports *v* and *c*:

$$X_B^k(v) = \int_0^1 x(v, \hat{c}) g^k(\hat{c}) d\hat{c}, \qquad X_S^k(c) = \int_0^1 x(\hat{v}, c) f^k(\hat{v}) d\hat{v},$$

$$(6.1) \qquad P_B^k(v) = \int_0^1 p(v, \hat{c}) g^k(\hat{c}) d\hat{c}, \qquad P_S^k(c) = \int_0^1 p(\hat{v}, c) f^k(\hat{v}) d\hat{v},$$

$$U_B^k(v) = vP_B^k(v) - X_B^k(v), \qquad U_S^k(c) = X_S^k(c) - cP_S^k(c).$$

For a given *k*, the mechanism *p*(·, ·), *x*(·, ·) is *Lk*-incentive-compatible if and only if truthful reporting is optimal given *Lk* beliefs; that is, iff for every *V*, *v*, *C*, and *c* in [0, 1],

$$(6.2) \qquad U_B^k(V) \geq VP_B^k(v) - X_B^k(v) \;\text{ and }\; U_S^k(C) \geq X_S^k(c) - CP_S^k(c).$$

The mechanism *p*(·, ·), *x*(·, ·) is (interim) *Lk*-individually rational iff for every *V* and *C* in [0, 1],

$$(6.3) \qquad U_B^k(V) \geq 0 \;\text{ and }\; U_S^k(C) \geq 0.$$

Theorems B and C extend MS's (Theorems 1-2) characterization of equilibrium-incentive-efficient mechanisms to level-*k* models with known, homogeneous populations of *L1s* or *L2s*, showing that in this case MS's characterization is qualitatively fully robust to level-*k* thinking.[13]

Theorem B. *For any known population of L1 or L2 traders concentrated on one level,* k, *and any level-k-incentive-compatible mechanism,*

$$(6.4) \qquad U_B^k(0) + U_S^k(1) = \min_{V\in[0,1]} U_B^k(V) + \min_{C\in[0,1]} U_S^k(C)$$

$$= \int_0^1 \int_0^1 \left(\left[V - \frac{1 - F(V)}{f(V)} \right] \left[\frac{g^k(C; p, x)}{g(C)} \right] - \left[C + \frac{G(C)}{g(C)} \right] \left[\frac{f^k(V; p, x)}{f(V)} \right] \right) p(V, C) f(V) g(C) dC dV.$$

Furthermore, if p(·,·) is any function mapping [0, 1]×[0, 1] *into* [0, 1], *there exists a function x(·,·) such that (p, x) is level*-k-*incentive-compatible and level*-k-*interim-individually rational if and only if* $P_B^k(\cdot)$ *is weakly increasing for all (p, x),* $P_S^k(\cdot)$ *is weakly decreasing for all (p, x), and*

[13] This robustness may be surprising because MS's analysis relies on the behaviorally strong assumption that traders play an equilibrium that is a fixed-point in a high-dimensional strategy space, while level-*k* models avoid fixed-point reasoning. Theorems B's and C's conclusions hold for any nonequilibrium model in which players have homogeneous decision rules that are continuous and best respond to some decoupled beliefs, as long as the corresponding monotonicity conditions are satisfied.

$$(6.5)\ \ 0 \le \int_0^1\int_0^1 \left\{\left[V - \frac{1-F(V)}{f(V)}\right]\left[\frac{g^k(C;p,x)}{g(C)}\right] - \left[C + \frac{G(C)}{g(C)}\right]\left[\frac{f^k(V;p,x)}{f(V)}\right]\right\} p(V,C)f(V)g(C)dCdV.$$[14]

Proof. The proof follows MS's, with adjustments for traders' nonequilibrium beliefs. By (6.1), $P_B^k(\cdot)$ is weakly increasing and $P_S^k(\cdot)$ is weakly decreasing for any given (*p*, *x*), which as in MS's proof (pp. 269-270) yields necessary and sufficient conditions for incentive-compatibility:

(6.6) $U_B^k(V) = U_B^k(0) + \int_0^V P_B^k(v)dv$ and $U_S^k(C) = U_S^k(1) + \int_C^1 P_S^k(c)dc$ for all V and C.

(6.6) implies that $U_B^k(V)$ is weakly increasing and $U_S^k(C)$ is weakly decreasing, and shows that $U_B^k(0) \ge 0$ and $U_S^k(1) \ge 0$ suffice for individual rationality for all V and C as in (6.3).[15]

To derive the incentive budget constraint (6.5), the level-*k* analogue of the equilibrium (3.5) or MS's (2), note that, given that each trader knows the true value densities and reports his value truthfully (but does not in general expect his partner to report truthfully),

$$\int_0^1\int_0^1 V\,p(V,\hat{c})g^k(\hat{c})d\hat{c}f(V)dV - \int_0^1\int_0^1 C\,p(\hat{v},C)f^k(\hat{v})d\hat{v}g(C)dC =$$

$$\int_0^1 U_B^k(V)f(V)dV + \int_0^1 U_S^k(C)g(C)dC =$$

$$(6.7)\qquad U_B^k(0) + \int_0^1\int_0^V P_B^k(v)dvf(V)dV + U_S^k(1) + \int_0^1\int_C^1 P_C^k(c)dcg(C)dC =$$

$$U_B^k(0) + U_S^k(1) + \int_0^1[1-F(v)]P_B^k(v)dv + \int_0^1 G(c)P_S^k(c)dc =$$

$$U_B^k(0) + U_S^k(1) + \int_0^1\int_0^1[1-F(v)]p(v,\hat{c})g^k(\hat{c})d\hat{c}dv + \int_0^1\int_0^1 G(c)p(\hat{v},c)f^k(\hat{v})d\hat{v}dc.$$[16]

Equating the first and last expressions in (6.7) (with a change of variables of integration) yields (6.4), which implies (6.5) when the mechanism is individually rational. And given (6.3) and that $P_B^k(\cdot)$ is weakly increasing and $P_S^k(\cdot)$ is weakly decreasing, MS's (pp. 270-271) transfer function

$$(6.8)\qquad x(v,c) = \int_0^V v\,d[P_B^k(v)] - \int_0^C c\,d[-P_S^k(c)] + \int_0^1 c[1-G(c)]\,d[-P_S^k(c)]$$

makes (*p*, *x*) level-*k*-incentive-compatible and level-*k*-interim individually rational. Q.E.D.

Theorem C, the level-*k* analogue of MS's Theorem 2, characterizes mechanisms that are efficient in the set of level-*k*-incentive-compatible mechanisms.

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[14] With correct beliefs, $g^k(C;p,x) \equiv g(C)$ and $f^k(V;p,x) \equiv f(V)$, (6.5) is equivalent to MS's (2) incentive budget constraint, (3.5). Because level-*k* beliefs happen to be correct for uniform value densities (for all *k*), that equivalence implies Theorem A.

[15] This characterization is possible in MS's equilibrium-based analysis because the revelation principle decouples the problems that determine whether truth-telling is a best response. Level-*k* best responses decouple even without the revelation principle.

[16] (6.4) and (6.5) "add apples and oranges" because the buyer's and the seller's beliefs differ. This is not a problem because *Lk* traders' incentive constraints decouple, and the transfer function compensates each trader according to his own beliefs.

Theorem C. *For any known population of L1 or L2 traders concentrated on one level, if there exists a mechanism (p, x) that is level*-k-*incentive-compatible and maximizes traders' ex ante expected total surplus* $\int_0^1 \int_0^1 (V - C)\, p(V, C) f(V) g(C) dCdV$ *s.t.* $U_B^k(0) = U_S^k(1) = 0$ *and (6.5), and*

(6.9) $$\left[V - \frac{1-F(V)}{f(V)}\right]\left[\frac{g^k(C;p,x)}{g(C)}\right] - \left[C + \frac{G(C)}{g(C)}\right]\left[\frac{f^k(V;p,x)}{f(V)}\right]$$

is increasing in V and decreasing in C for any given (p, x), then that mechanism is efficient in the set of level-k-*incentive-compatible and level*-k-*interim-individually-rational mechanisms.*

Proof. The proof adapts MS's proof of Theorem 2 (pp. 275-276). Consider the problem of choosing $p(\cdot, \cdot)$ to maximize ex ante expected total surplus subject to $0 \leq p(\cdot, \cdot) \leq 1$, $U_B^k(0) = U_S^k(1) = 0$, and (6.5). That problem is like a consumer's budget problem, with a continuum of trade probabilities $p(V, C)$, which are analogous to linearly priced goods.[17] Form the Lagrangean (for ease of notation, without separately pricing out the $p(V, C) \leq 1$ constraints):

$$\int_0^1 \int_0^1 (V - C)\, p(V, C) f(V) g(C) dCdV$$

(6.10) $$+\lambda \int_0^1 \int_0^1 \left\{\left[V - \frac{1-F(V)}{f(V)}\right]\left[\frac{g^k(C;p,x)}{g(C)}\right] - \left[C + \frac{G(C)}{g(C)}\right]\left[\frac{f^k(V;p,x)}{f(V)}\right]\right\} p(V, C) f(V) g(C) dCdV$$

$$= \int_0^1 \int_0^1 \left((V - C) + \lambda\left\{\left[V - \frac{1-F(V)}{f(V)}\right]\left[\frac{g^k(C;p,x)}{g(C)}\right] - \left[C + \frac{G(C)}{g(C)}\right]\left[\frac{f^k(V;p,x)}{f(V)}\right]\right\}\right) p(V, C) f(V) g(C) dCdV.$$

The objective function and the constraint are linear in the $p(V, C)$, so the solution is "bang-bang", with $p(V, C) = 0$ or 1 almost everywhere. The Kuhn-Tucker conditions require $\lambda \geq 0$,

(6.11) $(V - C) + \lambda\left\{\left[V - \frac{1-F(V)}{f(V)}\right]\left[\frac{g^k(C;p,x)}{g(C)}\right] - \left[C + \frac{G(C)}{g(C)}\right]\left[\frac{f^k(V;p,x)}{f(V)}\right]\right\} \leq 0$ when $p(V, C) = 0$, and

(6.12) $(V - C) + \lambda\left\{\left[V - \frac{1-F(V)}{f(V)}\right]\left[\frac{g^k(C;p,x)}{g(C)}\right] - \left[C + \frac{G(C)}{g(C)}\right]\left[\frac{f^k(V;p,x)}{f(V)}\right]\right\} \geq 0$ when $p(V, C) = 1$.

In the analogy, (6.11)-(6.12) say that it is optimal to buy the goods with the highest (positive) marginal-utility-to-price ratios, until the incentive budget constraint (6.5) binds. Because *V* and *C* are continuously distributed, (6.5) holds with equality at the solution. If (6.9) is increasing in *V* and decreasing in *C* for any (*p*, *x*), then $p(V, C)$ and thus $P_B^k(V)$ and $P_S^k(C)$ in (6.1) are respectively increasing and decreasing. Then (and generally only then) by Theorem B, the solution of the problem is associated with a mechanism that maximizes expected total surplus among all level-*k*-incentive-compatible and level-*k*-individually-rational mechanisms. Q.E.D.

[17] Some of the prices are negative; but by the logic of the incentive constraints, there is no free disposal. A solution exists (even if $k = 2$, despite the two-way recursion between (*p*, *x*) and $f^2(V; p, x)$ and $g^2(C; p, x)$) because continuity of the value densities ensures continuity of the objective function and the constraint function, and the feasible region is compact.

Theorem C's condition that (6.9) is increasing in V and decreasing in C for all (*p, x*) is the level-*k* analogue of MS's (Theorem 2) equilibrium condition for $p(V,C) = 1$ that (in my (3.8)'s notation) both $\Phi(V,1)$ and $\Gamma(C,1)$ are increasing on [0, 1]. Theorem C's condition on (6.9) with the true densities $f(V)$ and $g(C)$ replacing the level-*k* beliefs $f^k(V;p,x)$ and $g^k(C;p,x)$ reduces to MS's condition that $\Phi(V,1) - \Gamma(C,1)$ is increasing in *V* and decreasing in *C*. That condition is satisfied when the true densities fit into Myerson's (1981) "regular case", which rules out strong hazard rate variations in the wrong direction. Theorem C's level-*k* version of the condition, on (6.9), jointly restricts the true densities and level-*k* beliefs in a similar way.

Comparing the level-*k* incentive budget constraint (6.5) with MS's equilibrium incentive budget constraint (MS's (2); my (3.5)) and comparing the level-*k* Kuhn-Tucker condition (6.12) with the equilibrium-based condition (MS, p. 274; my (3.8)) shows that the design features that foster equilibrium-incentive-efficiency in MS's analysis also foster efficiency in the set of level-*k*-incentive-compatible mechanisms, although level-*k* beliefs give them different weights.

However, there are important differences in the level-*k* analysis. First, it is possible that the optimal $\lambda = 0$, so that from (6.12) $p(V,C) = 1$ iff $V \geq C$ (ignoring ties); (6.5) is satisfied then; and the mechanism that is efficient in the set of level-*k*-incentive-compatible mechanisms is ex post efficient with probability 1, contra MS's Corollary 1. To see this, adapt MS's proof of Corollary 1 (pp. 271-273) for a population concentrated on level *k*. With value densities supported on [0, 1] and $p(V,C) \equiv 1$ iff $V \geq C$, the incentive budget constraint (6.5) reduces to:

$$0 \leq \int_0^1 \int_0^1 \left\{\left[V - \frac{1-F(V)}{f(V)}\right]\left[\frac{g^k(C;p,x)}{g(C)}\right] - \left[C + \frac{G(C)}{g(C)}\right]\left[\frac{f^k(V;p,x)}{f(V)}\right]\right\} p(V,C) f(V) g(C) dC dV$$

$$= \int_0^1 \int_0^V [Vf(V) + F(V) - 1] g^k(C;p,x) dC dV - \int_0^1 \int_0^V [Cg(C) + G(C)] dC f^k(V;p,x) dV$$

$$(6.13) \quad = \int_0^1 [Vf(V) + F(V) - 1][G^k(V;p,x) - G^k(0;p,x)] dV - \int_0^1 VG(V) f^k(V;p,x)] dV$$

$$= -\int_0^1 [1 - F(V)] G^k(V;p,x) dV + \int_0^1 Vf(V) G^k(V;p,x) dV - \int_0^1 VG(V) f^k(V;p,x) dV.$$

With equilibrium beliefs, the last two terms on the last line exactly offset each other as in MS's proof, and the fact that the first term is negative proves MS's Corollary 1. However, for level-*k* beliefs, in extreme cases the middle term could in theory outweigh the others, allowing perfect ex post efficiency. (I have been unable to find a tractable example to illustrate this possibility.)

Second, there is another design feature that fosters efficiency in the set of level-*k*-incentive-compatible mechanisms, with no counterpart in the equilibrium analysis. Unless such a mechanism happens to induce correct beliefs (as with uniform value densities by Theorem A), it must benefit from tacit exploitation of predictably incorrect beliefs ("TEPIB"): "predictably incorrect" in that the level-k model predicts traders' deviations from equilibrium; "exploitation" in the benign sense of using traders' nonequilibrium responses for their own benefit; and "tacit" in that the mechanism does not actively mislead traders.

A thought-experiment clarifies the influence of TEPIB: Suppose one could exogenously increase the pessimism of traders' level-k beliefs relative to the truth, in the sense of first-order stochastic dominance (moving probability mass higher in the seller's beliefs $f^k(V;p,x)$ and/or lower in the buyer's beliefs $g^k(C;p,x)$). Then substituting (6.1) into (6.6) shows that, other things equal, that would loosen the incentive budget constraint (6.5). Because traders' beliefs enter the problem only through (6.5), that would increase maximized expected total surplus.

In the model it is not possible to exogenously change traders' beliefs, but the tradeoffs in (6.5) reflect the influence of traders' pessimism or optimism. Relative to the equilibrium incentive-efficient mechanism, TEPIB favors trade at (V, C) combinations for which $\frac{f^k(V;p,x)}{f(V)} > 1$ and/or $\frac{g^k(C;p,x)}{g(C)} < 1$, so that traders' non-equilibrium beliefs make the "prices" in curly brackets in (6.11)-(6.12) more favorable. And for k = 2 (only, because *L1*s' beliefs do not depend on the mechanism), TEPIB also favors mechanisms that increase the advantages of such trades.

Note however that, even if both traders' level-k beliefs are more pessimistic than equilibrium beliefs, so their maximized expected total surplus is higher, the possibility of negative prices in (6.5) means that a mechanism that is efficient in the set of level-k-incentive-compatible mechanisms might not have a trading region uniformly larger than that of an equilibrium-incentive-efficient mechanism. (The examples in Figure 1 generally confirm the intuition that pessimism favors more efficient trading, but also reflect this possibility.)

Finally, Theorem C and the Kuhn-Tucker condition (6.12) show that a mechanism that is efficient in the set of level-k-incentive-compatible mechanisms may involve trade at some value combinations with $V < C$: "perverse" ex post, though consistent with level-k-interim-individual-rationality. By MS's Theorem 2 (my (3.8)), ex-post-perverse trade cannot occur in an equilibrium-incentive-efficient mechanism, although MS note (p. 271) that their transfer function sometimes violates ex-post individual rationality by requiring payment from buyers who do not get the object. In the level-k analysis some of the prices in the incentive budget constraint (6.5) are negative (with no free disposal; fn 17). That makes (6.12) consistent with some ex-post-perverse trade, which can loosen (6.5) enough to compensate for the local loss in surplus by enabling trade for other value combinations. (Figure 1's examples confirm that this can happen.)

As in MS's analysis, closed-form solutions are possible only for uniform value densities; but for them the mechanism that is efficient in the set of level-*k*-incentive-compatible mechanisms induces correct beliefs (Theorem A), so that TEPIB has no influence. To illustrate TEPIB's influence, Figure 1 reports the trading regions for mechanisms that are efficient in the set of *L1*-incentive-compatible mechanisms for representative combinations of linear value densities, with the regions for the corresponding equilibrium-incentive-efficient mechanisms for comparison.[18]

Such mechanisms are similar to equilibrium-incentive-efficient mechanisms in most respects. For true (e.g. upward-sloping) densities that make *L1* traders' uniform beliefs pessimistic, mechanisms that are efficient in the set of *L1*-incentive-compatible mechanisms usually have trading regions that are supersets of equilibrium-incentive-efficient mechanisms regions, with overlapping regions for a few such combinations (e.g. "0.75, 1.75").[19] By contrast, for true (e.g. downward-sloping) densities that make *L1* traders' beliefs optimistic, TEPIB still has an influence, but the equilibrium-incentive-efficient trading regions are usually supersets of the regions for *L1*s (with some overlaps, e.g. "1.0, 0.25", "1.5, 0.25", and "1.5, 0.50"). In two combinations with densities that are extreme in opposite directions ("0.25, 1.5" and "0.25, 1.75"), *L1* mechanisms require ex post perverse trade for very high values of *V* and *C*.

6.4 Mechanisms that are efficient in the set of level-k-incentive-efficient mechanisms with general value densities and heterogeneous levels

This section discusses relaxing the assumption that the population is concentrated on one level, continuing to require level-*k*-incentive-compatibility, and assuming that there is a known mixture of *L1* and *L2* traders. I restrict attention to a single direct mechanism on the grounds that there is no evidence on which to base a specification of a level-*k* model for more complex menus. Suppose for the sake of argument (with heterogeneous levels it may not follow from optimization) that level-*k* incentive-efficient mechanisms set $U_B^k(0) = U_S^k(1) = 0$ for *k* = 1, 2.

[18] The density combinations are a comprehensive coarse subset of all possible linear density combinations, with a few extreme combinations excluded because they violate the monotonicity conditions that, by Theorems B-C, are needed for the mechanism to be truly optimal. The computations are infeasible for *L2*s, because with $f^2(v) \equiv f(\beta_1^{-1}(v;p,x))$ and $g^2(c) \equiv g(\alpha_1^{-1}(c;p,x))$, (6.5) and (6.12) depend on the transfer function $x(\cdot,\cdot)$ as well as on $p(\cdot,\cdot)$, and the dimensionality of search is too high. The online appendix provides the MATLAB code for *L1*s, written by Rustu Duran.

[19] Surprisingly, pessimism for *L1* sellers tends to be somewhat more beneficial than pessimism for *L1* buyers. Because trade does not always occur when *V* > C, initial ownership of the object breaks the symmetry between buyer and seller; and even (3.5) and the associated equilibrium trading regions are not symmetric across cases with buyer's and seller's densities interchanged.

Even then, conditions like (6.5) require different transfers for different levels, but traders of any level would select the higher transfer, so only trivial mechanisms can completely screen traders' values and levels.[20] As a result, complete screening is normally suboptimal.

If the population contains mostly *L1*s (respectively *L2*s), it is likely that mechanisms that are efficient in the set of level-*k*-incentive-compatible mechanisms are optimized for *L1*s (*L2*s) as in Section 6.3, ignoring the rarer level but still getting some expected surplus from it. For less extreme level distributions, screening levels interacts with screening values, and I can identify no structure on which levels and values should be screened. I leave that for future work.

6.5 Mechanisms that are level-k-incentive-efficient, relaxing level-k-incentive-compatibility to allow any direct mechanism, with known populations concentrated on one level

This section returns briefly to a known population of *L1* or *L2* traders concentrated on one level, while relaxing the assumption that only level-*k*-incentive-compatible mechanisms are feasible, instead allowing any direct mechanism, incentive-compatible or not (see footnote 6).

Here one can still define a general class of feasible direct mechanisms, with payoff-relevant outcomes $p(v, c)$ and $x(v, c)$. However, a direct mechanism's incentive effects can no longer be tractably captured via constraints like (6.2) and (6.6), but must be modeled directly via level-*k* traders' responses to it. I call a mechanism "level-*k*-incentive-efficient" if its outcomes cannot be improved upon by any feasible direct mechanism, given traders' level-*k* responses.

As a tractable proxy for what is theoretically achievable via any feasible direct mechanism, I focus on double auctions with reserve prices chosen by the designer, assuming uniform value densities. Reserve prices have no benefits if *Lk* traders continue to anchor their beliefs with *L0* uniform random on the full range of possible values [0, 1]. But a double auction with a restricted menu of bids or asks may make *Lk* traders anchor on the restricted menu instead of [0, 1], and such anchoring can make reserve prices useful in trading mechanisms.[21]

For example, in the double auction with uniform value densities, *L1* traders believe they face bids or asks uniformly distributed on [0, 1], which leads to *L1*-incentive-*in*efficient outcomes. To implement the outcome of MS's equilibrium-incentive-efficient direct mechanism via the double auction, *L1* traders must believe that they face bids or asks uniform on [1/4, 3/4], the range of

[20] I treat the differences in traders' levels as pure differences of opinion, as in Eliaz and Spiegler (2008): Traders neither believe their partners are better or worse informed, nor draw inferences about levels from each other's decision or the mechanism.

[21] I know of no evidence for or against such an *L0* specification, but in marketing such menu effects are commonplace. Crawford et al. (2009) showed that in auctions, such anchoring can make reserve prices useful even if they are useless with equilibrium bidders. Saran (2011b) studies how menu-dependent preferences affect the revelation principle.

“serious” bids or asks in CS’s linear double-auction equilibrium. If *L1* traders anchor on the restricted menu, those beliefs can be induced by restricting bids to [1/4, 3/4] and asks to [1/4, 3/4].[22] Thus with uniform value densities, for *L1*s a double auction with reserve prices can use TEPIB to mimic the outcomes of MS’s equilibrium-incentive-efficient mechanism, whose direct form is then efficient in the set of *L1*-incentive-compatible mechanisms.[23] Tedious computations suggest that more stringent reserve prices can further improve upon the mechanism that is efficient in the set of *L1*-incentive-compatible mechanisms, by taking fuller advantage of TEPIB.

For *L2*s with uniform value densities, Section 5’s analysis already shows that a double auction without reserve prices can improve upon a mechanism that is efficient in the set of *L2*-incentive-efficient mechanisms, or upon MS’s equilibrium-incentive-efficient mechanism. Again, computations suggest that reserve prices allow even more improvement, via TEPIB.

More generally, relaxing the restriction to level-*k*-incentive-compatible mechanisms can yield level-*k*-incentive-efficient mechanisms that differ qualitatively as well as quantitatively from equilibrium-incentive-efficient mechanisms, with substantial gains in incentive-efficiency.

7. CONCLUSION

The level-*k* model I use to study mechanism design for bilateral trading makes predictions specific enough to allow an analysis with power comparable to an equilibrium analysis. The results clarify the role of MS’s equilibrium assumption in several ways. First, the choice of mechanism has menu effects that influence the correctness of level-*k* beliefs. Sometimes such effects allow direct mechanisms that are not level-*k*-incentive-compatible (in the narrow truthful-revelation sense) such as the double auction, if feasible, to yield more efficient outcomes than those achievable via any level-*k*-incentive-compatible mechanism; and sometimes vice versa.

Either way, it matters whether level-*k*-incentive-compatibility is required. If it is, MS’s result that with uniform value densities, the equilibrium-incentive-efficient direct mechanism mimics CS’s linear double-auction equilibrium, is completely robust to replacing equilibrium with a level-*k* model. Further, for known populations concentrated on one level, MS’s characterization of incentive-efficient mechanisms for general value densities is fully robust to level-*k* thinking. As a result, the design features that foster equilibrium-incentive-efficiency in MS’s analysis also

[22] The upper limit 3/4 of asks could be raised to 1 without altering an *L1* buyer’s bids, and the lower limit 1/4 of bids could be lowered to 0 without altering an *L1* seller’s asks.

[23] MS’s general specification of feasible mechanisms implicitly allows reserve prices, and their analysis therefore shows that if equilibrium is assumed, reserve prices are not useful in this setting.

foster incentive-efficiency in the level-*k* analysis, although level-*k* beliefs give them different weights. The level-*k* analysis also reveals another design feature that fosters incentive-efficiency, TEPIB (tacit exploitation of predictably incorrect beliefs), whereby mechanisms that are efficient in the set of level-*k*-incentive-compatible mechanisms exploit traders' nonequilibrium beliefs, in the benign sense of implementing outcomes that increase their welfares, without deception.

Computations suggest that mechanisms that are efficient in the set of *L1*-incentive-compatible mechanisms are similar to equilibrium-incentive-efficient mechanisms in most respects. When the true value densities make *L1* traders' uniform beliefs pessimistic (that is, when the buyer's true density is upward-sloping and/or the seller's is downward-sloping), then such mechanisms use TEPIB to implement trading regions that are usually supersets of those of equilibrium-incentive-efficient mechanisms. When the true densities make *L1* traders' beliefs optimistic, such mechanisms still use TEPIB, but implement trading regions that are usually subsets of those of equilibrium-incentive-efficient mechanisms. Finally, for some extreme value densities, such mechanisms require ex-post-perverse trade for high values of V and C.

However, despite the theoretical possibility that level-*k* anchoring on a uniform *L0* could reduce incentive-efficient mechanisms' sensitivity to distributional and knowledge assumptions, as advocated by Wilson (1987), mechanisms that are efficient in the set of level-*k*-incentive-compatible mechanisms are just as sensitive as equilibrium-incentive-efficient mechanisms.

The level-*k* analysis yields some examples of non-robustness of the equilibrium analysis. Even if level-*k*-incentive-compatibility is required, MS's Corollary 1, that no incentive-compatible, interim individually rational mechanism is ex-post efficient with probability one, need not extend to level-*k* models. Sorting traders' levels along with their values poses new and formidable analytical problems. And if *non*-level-*k*-incentive-compatible direct mechanisms are feasible, then level-*k*-incentive-efficient mechanisms may differ qualitatively from equilibrium-incentive-efficient mechanisms, with the theoretical possibility of large efficiency gains.

It is my hope that this paper's analysis, incomplete as it is, will show that further progress in mechanism design is possible without assuming equilibrium, and so encourage further study.

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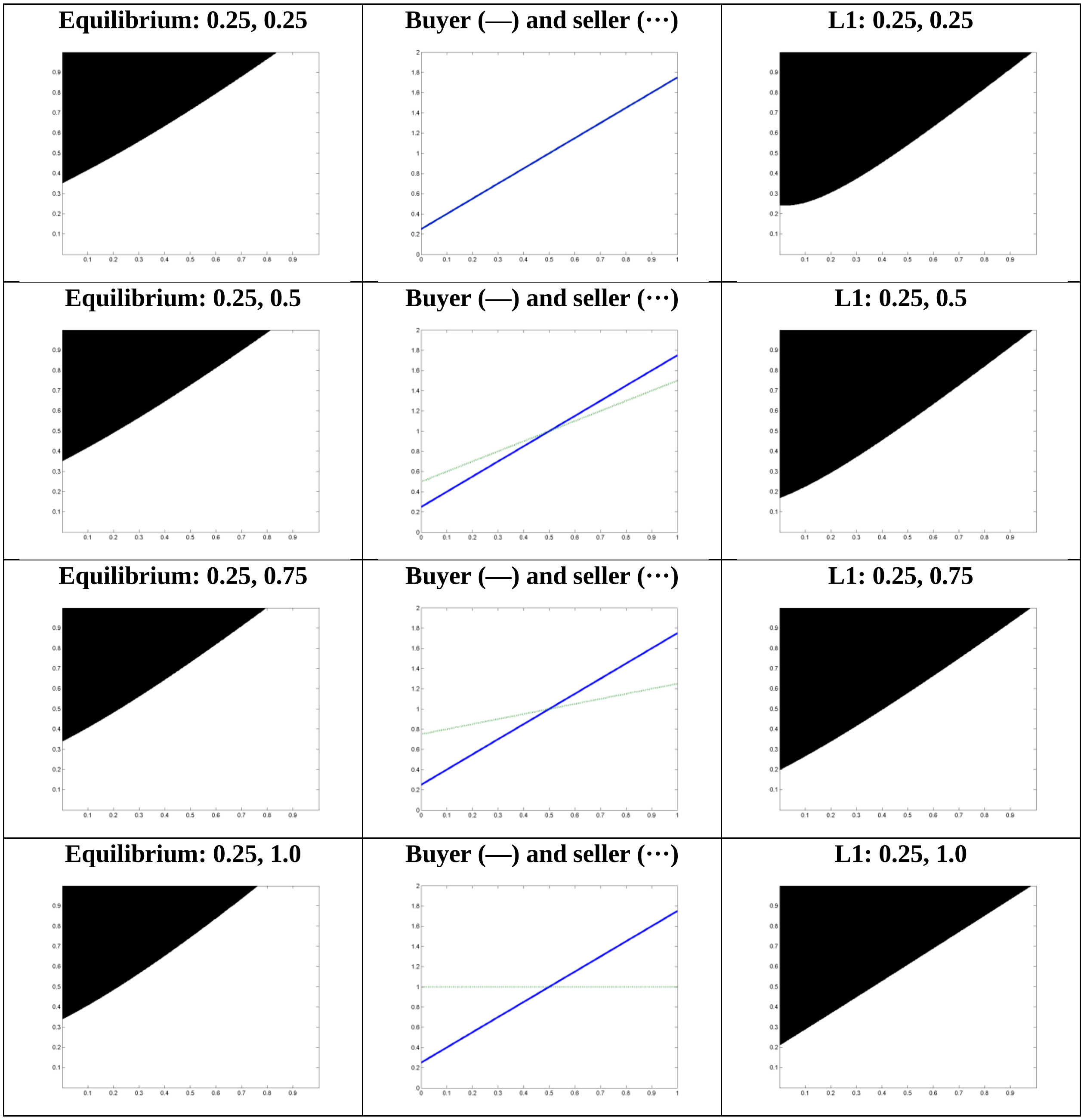
Figure 1. Trading regions (in black) for equilibrium-incentive-efficient mechanisms and mechanisms that are efficient in the set of *L1*-incentive-compatible mechanisms[24]



[24] Buyer's value *V* is on the vertical axis; seller's value *C* is on the horizontal axis. All value densities are linear; "*x*, *y*" means the buyer's density *f*(*V*) satisfies *f*(0) = *x* and *f*(1) = 2-*x*, and the seller's density *g*(*C*) satisfies *g*(0) = *y* and *g*(1) = 2-*y*.

Figure 1 (cont.). Trading regions (in black) for equilibrium-incentive-efficient mechanisms and mechanisms that are efficient in the set of *L1*-incentive-compatible mechanisms

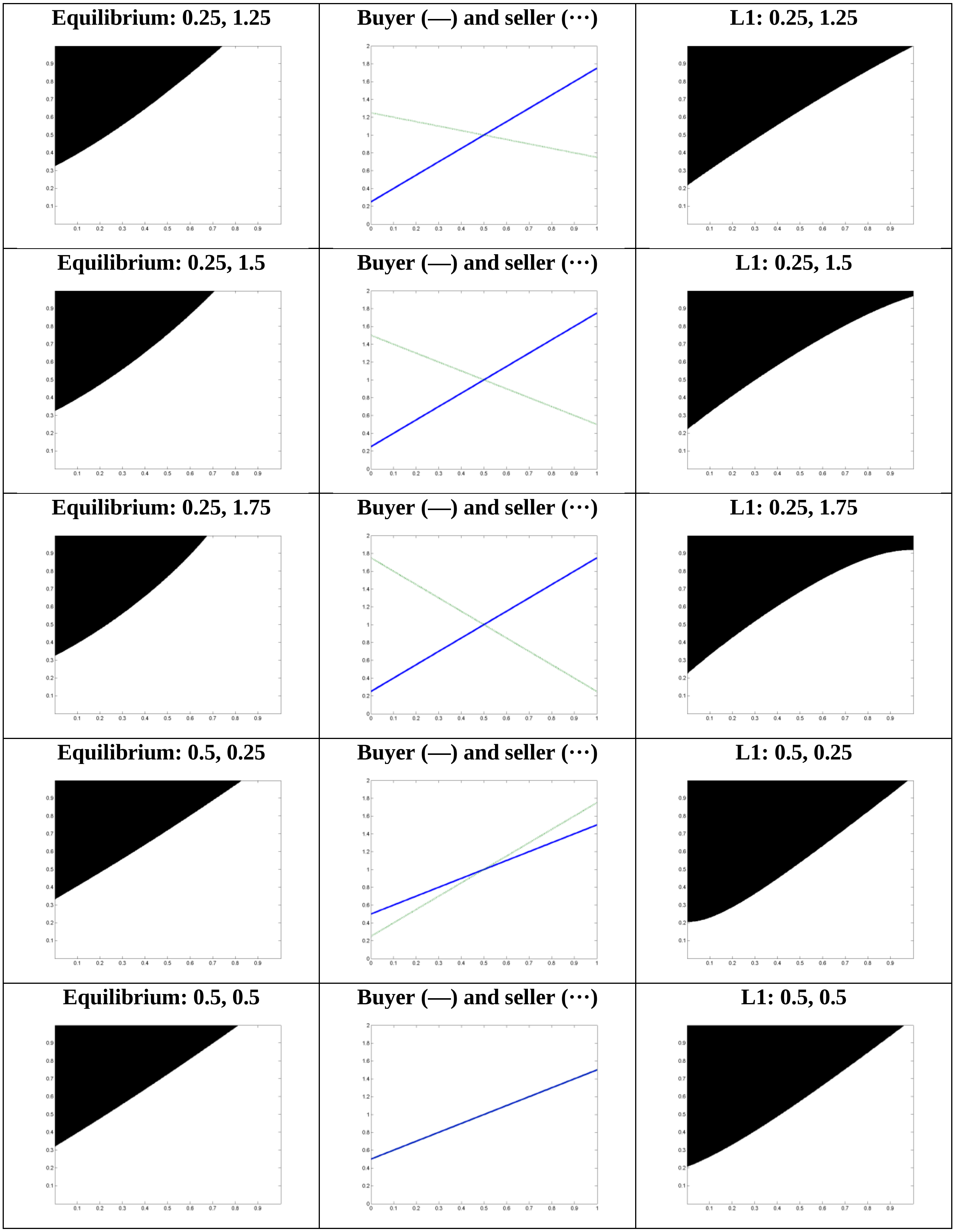


Figure 1 (cont.). Trading regions (in black) for equilibrium-incentive-efficient mechanisms and mechanisms that are efficient in the set of *L1*-incentive-compatible mechanisms

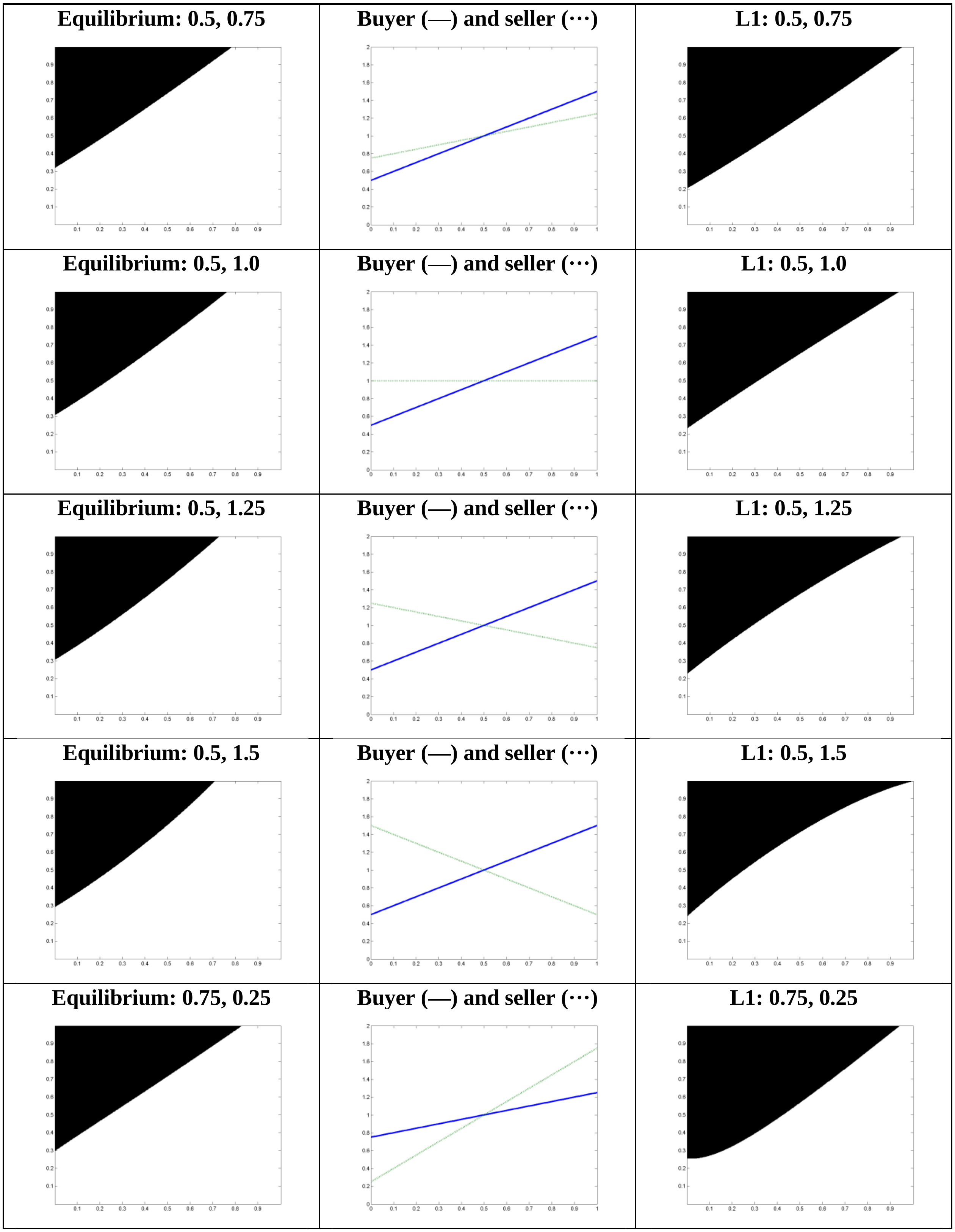


Figure 1 (cont.). Trading regions (in black) for equilibrium-incentive-efficient mechanisms and mechanisms that are efficient in the set of *L1*-incentive-compatible mechanisms

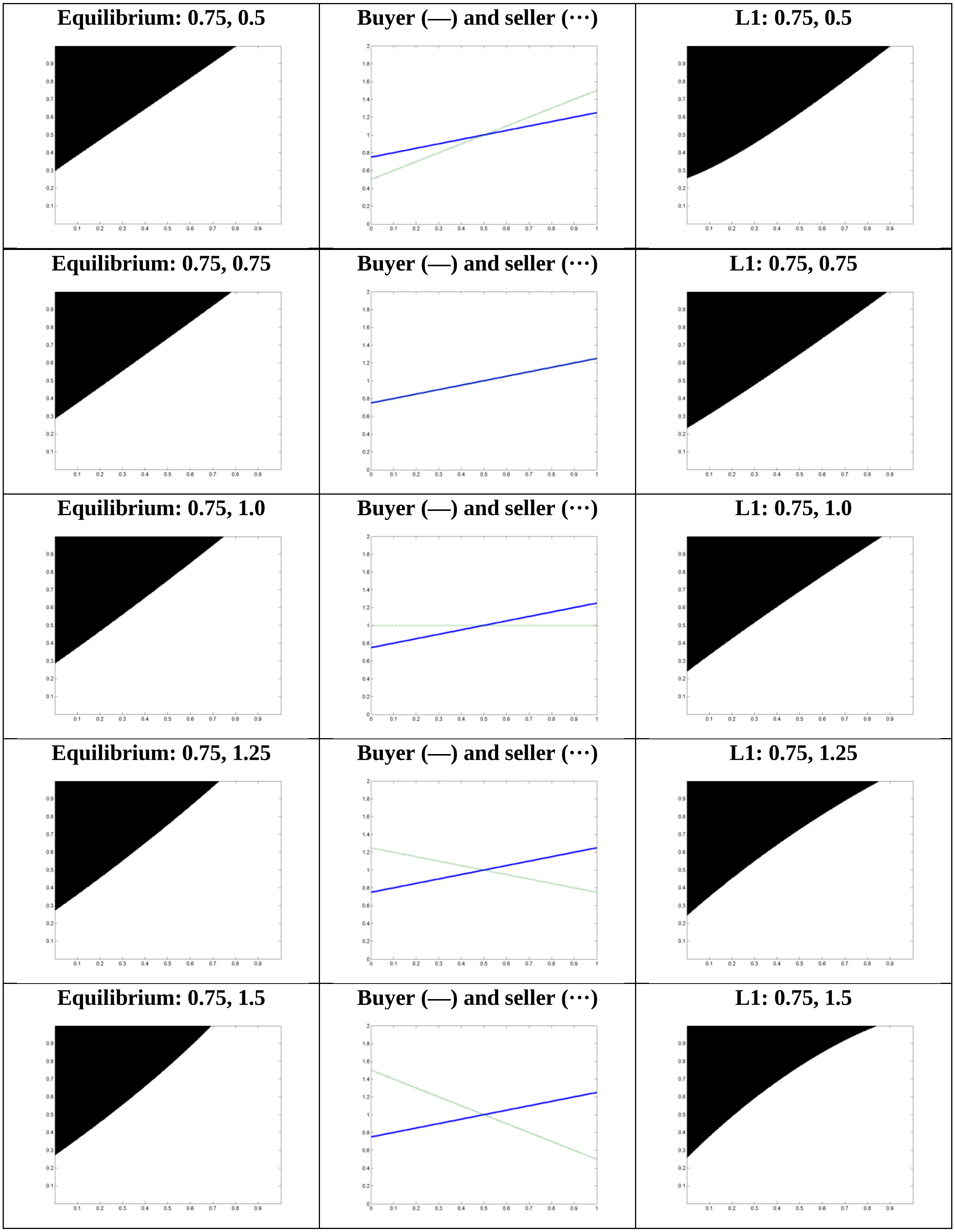


Figure 1 (cont.). Trading regions (in black) for equilibrium-incentive-efficient mechanisms and mechanisms that are efficient in the set of *L1*-incentive-compatible mechanisms

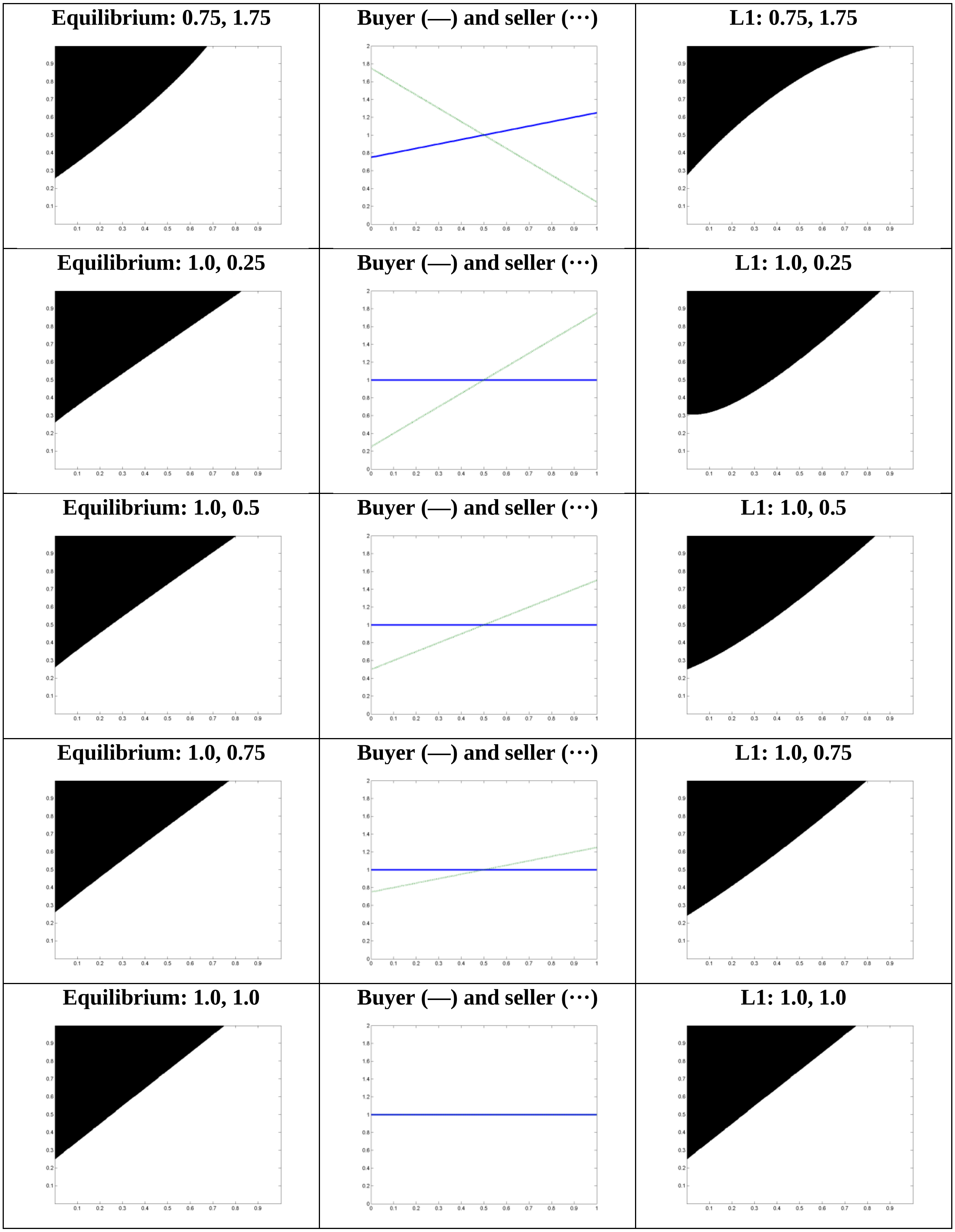


Figure 1 (cont.). Trading regions (in black) for equilibrium-incentive-efficient mechanisms and mechanisms that are efficient in the set of *L1*-incentive-compatible mechanisms

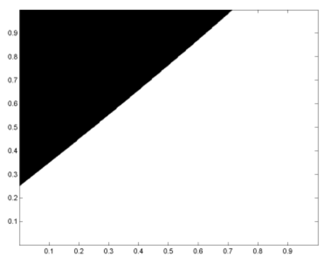
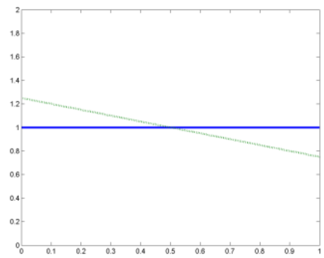
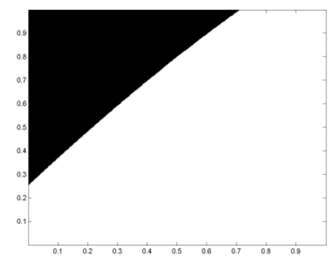
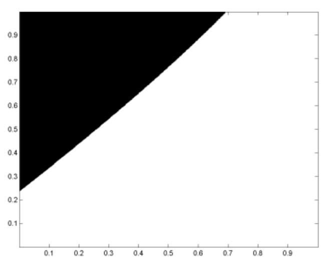
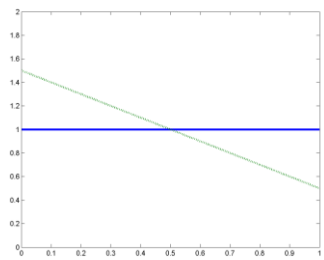
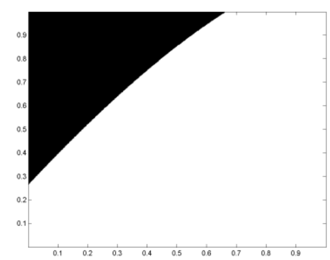
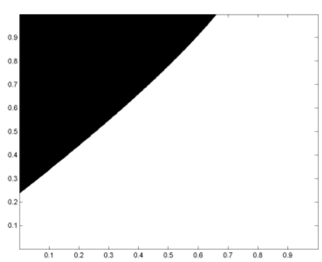
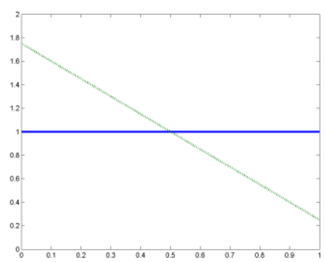
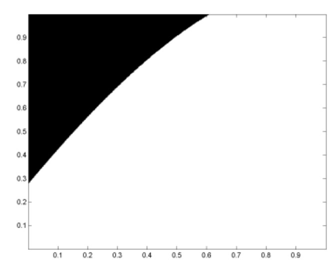
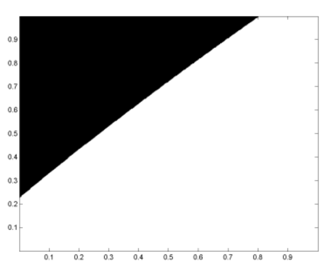
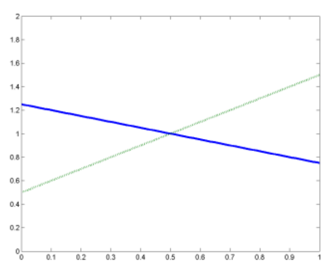
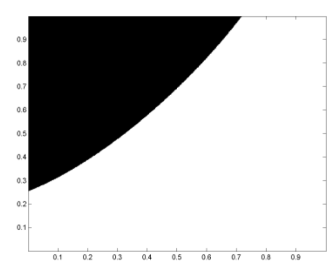
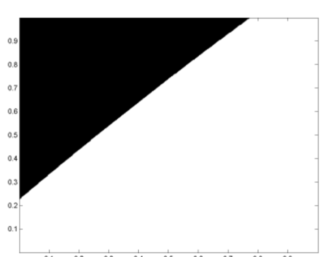
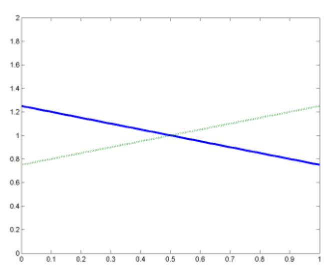
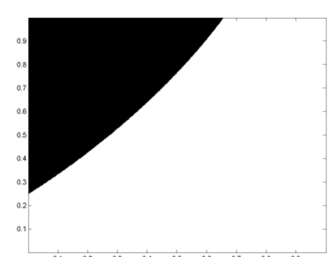
Equilibrium: 1.0, 1.25 	**Buyer (—) and seller (···)** 	**L1: 1.0, 1.25** 
Equilibrium: 1.0, 1.5 	**Buyer (—) and seller (···)** 	**L1: 1.0, 1.5** 
Equilibrium: 1.0, 1.75 	**Buyer (—) and seller (···)** 	**L1: 1.0, 1.75** 
Equilibrium: 1.25, 0.5 	**Buyer (—) and seller (···)** 	**L1: 1.25, 0.5** 
Equilibrium: 1.25, 0.75 	**Buyer (—) and seller (···)** 	**L1: 1.25, 0.75** 

Figure 1 (cont.). Trading regions (in black) for equilibrium-incentive-efficient mechanisms and mechanisms that are efficient in the set of *L1*-incentive-compatible mechanisms

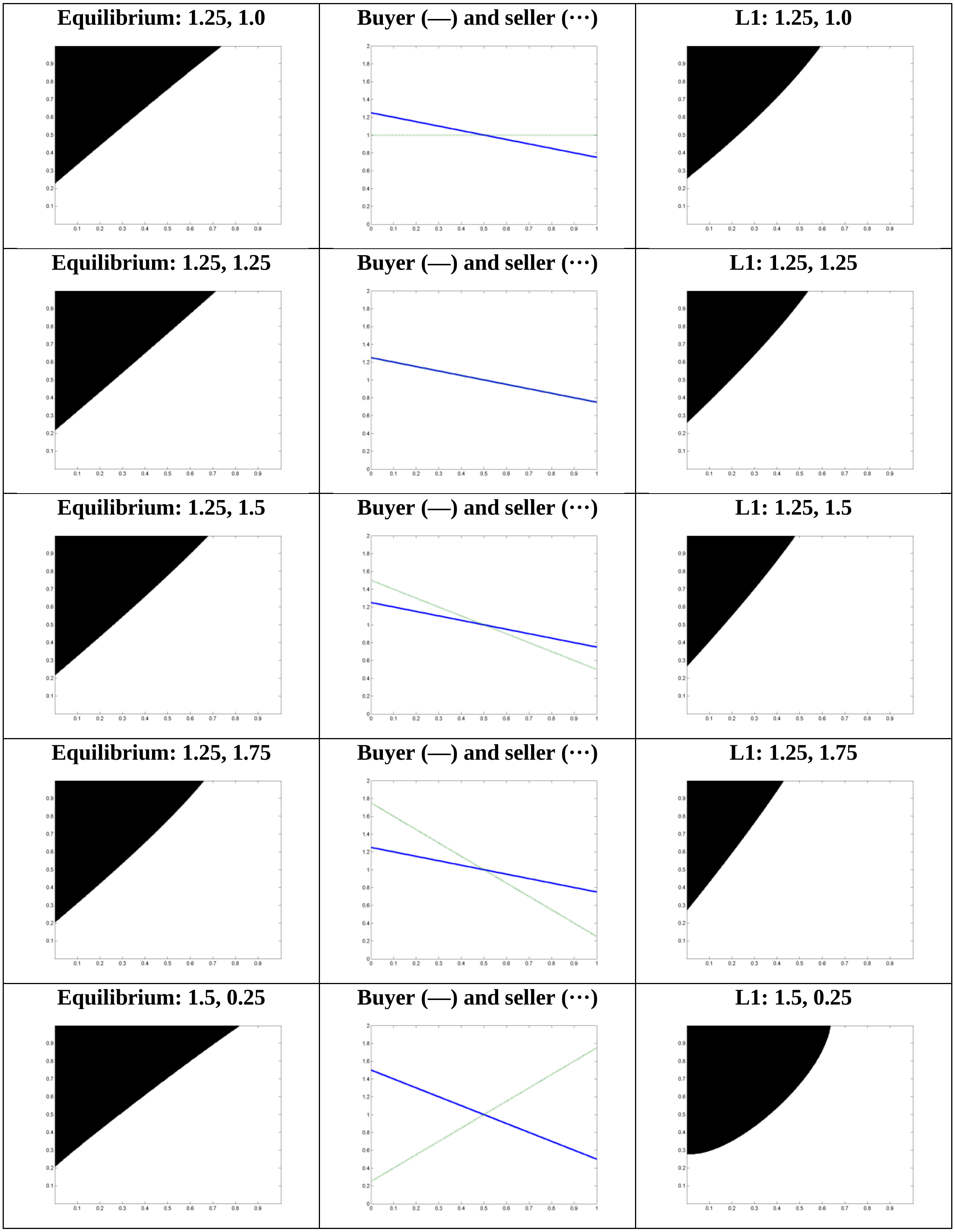


Figure 1 (cont.). Trading regions (in black) for equilibrium-incentive-efficient mechanisms and mechanisms that are efficient in the set of *L1*-incentive-compatible mechanisms

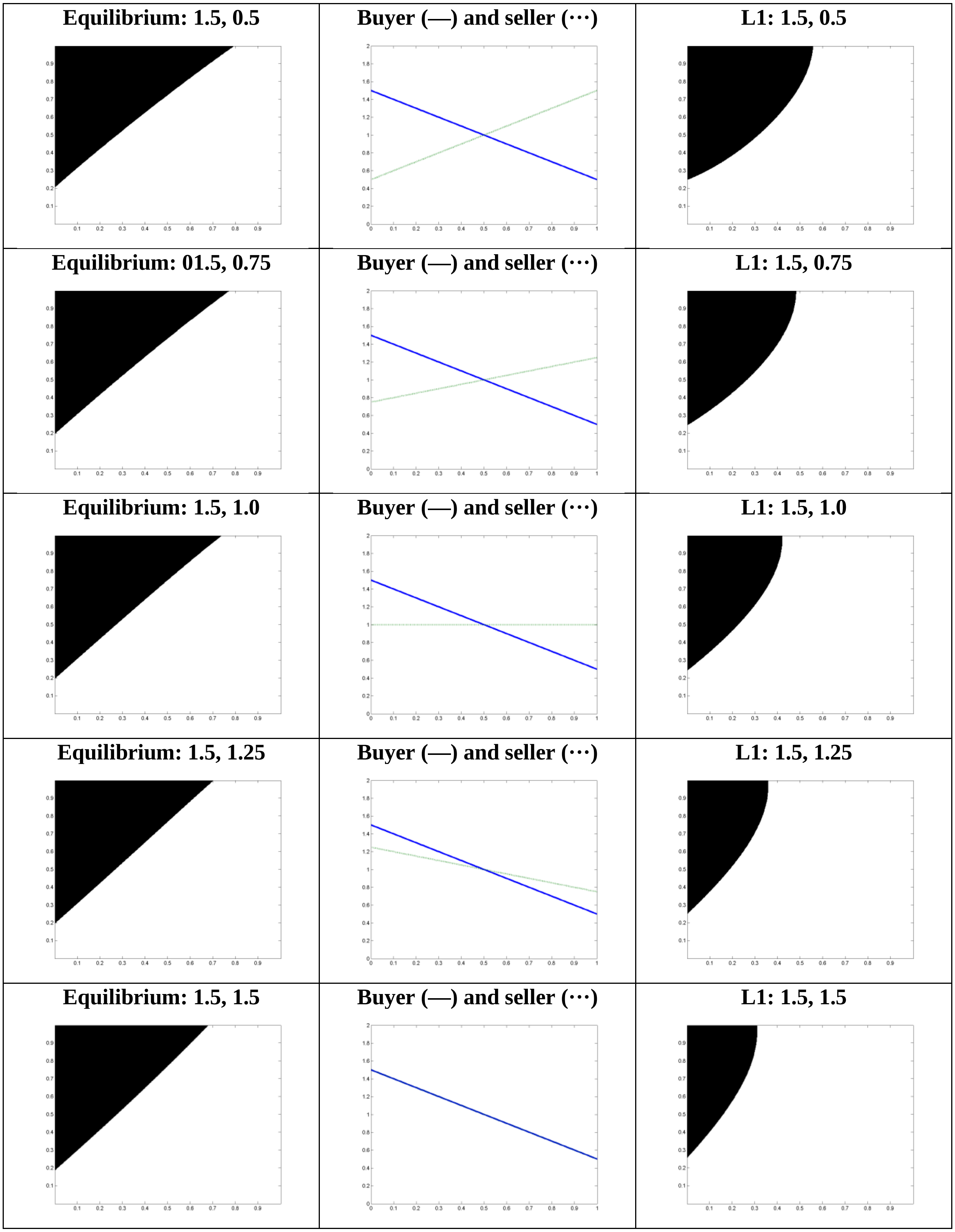
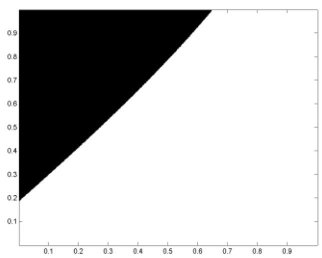
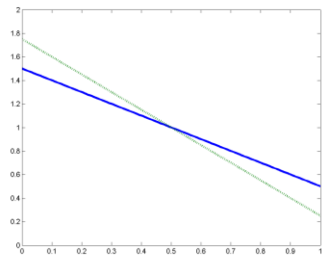
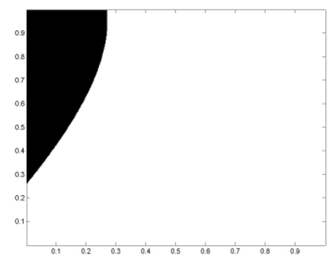


Figure 1 (cont.). Trading regions (in black) for equilibrium-incentive-efficient mechanisms and mechanisms that are efficient in the set of *L1*-incentive-compatible mechanisms

Equilibrium: 1.5, 1.75	Buyer (—) and seller (···)	L1: 1.5, 1.75
		

Online appendix: MATLAB code, developed by Rustu Duran, University of Oxford, for computation of mechanisms that are efficient in the set of *L1*-incentive-compatible mechanisms with a homogenous *L1* population, or that are equilibrium incentive-efficient.

The code assumes a homogeneous population of *L1*s and full-support linear densities of values *V* and *C*, each characterized by the value of *f*(0) or *g*(0), which, given linearity, range from 0 to 2, with *f*(0) = 1 or *g*(0) = 1 corresponding to uniform densities. *f*(0) and *g*(0) are represented by categorical index variables called fbar and gbar as follows. The interval [0, 2] is discretized into nine points, 0, 0.25, 0.50,…, 2.0, with index *i* representing the ith point. For instance, fbar = 2 means *f*(0)= 0.25 and gbar = 3 means *g*(0) = 0.5.

Solution algorithm

For *L1*s, the algorithm fixes a pair of value densities. For each given value of λ ("alfa" in the code) starting from 0.05, the code first uses the Kuhn-Tucker condition (6.12) to determine for which (*v*, *c*) combinations *p*(*v*, *c*) = 1. It then integrates the incentive budget constraint (6.5) (with $f^k(v; p, x) \equiv 1$ and $g^k(c; p, x) \equiv 1$ for *L1*s) for that λ. It then iterates these operations, increasing λ by increments of 0.05, until it finds the λ that makes the value on the right-hand side of (6.5) smallest; and checks how often that value changes sign (never more than once in the calculations). Finally, it chooses the value of λ that makes the value of the right-hand side as close to 0 as possible from above. This entire operation is done separately for pair of value densities. Figure 1 is based on all possible discretized combinations of linear value densities.

For equilibrium traders, the algorithm works in a completely analogous fashion.

Using the code to implement the algorithm

To implement the algorithm, first run the program main.m.

surf(exanteprobtrade) then shows how the ex ante probability of trade varies with the indices fbar (on the left axis) and gbar (on the right).

surf(expectedtotalsurplus) shows how the expected total surplus varies with fbar and gbar.

blackandwhite(fbar, gbar, pi) shows the trading region, with the area where *p*(*v*, *c*) = 1 in black.

To do the analogous computations for equilibrium traders, add "s" to the end of the arguments; e.g. surf(exanteprobtrades) instead of surf(exanteprobtrade).

sidebyside(fbar1,gbar1,pi,pis,fbar,gbar) shows both the equilibrium and *L1* trading regions.

comparetradearea (fbar1,gbar1,fbar2,gbar2,pi) compares the two trade areas, with a value of 0 meaning no trade in either case; 1 (2) only in the second (first) case; and 3 in both cases.

blackandwhites.m

```
%this is the function for visualising trade zone in eqm model

function blackandwhite= blackandwhites (fbar,gbar,pis)

figure;
minuspis=1-pis(:,:,fbar,gbar);

blackandwhite=imagesc(minuspis);

set( gca,'XTick',0:50:450,'XTickLabel',{'0','0.1', '0.2','0.3', '0.4',...
    '0.5',  '0.6', '0.7', '0.8', '0.9', '1'},'YTick',0:50:450,...
    'YTickLabel',{'1','0.9','0.8', '0.7','0.6', '0.5', '0.4',  '0.3',...
    '0.2', '0.1'} );

colormap gray;

end
```

comparetradearea.m

```
% this function compares the trading areas.
% if the value of the function is 3, then it is a common trading area.
% if 2, then the first one trades but not the second one.
% if 1, only the second one trades. if 0, nobody trades.

function comparetradeareas = comparetradearea( fbar1,gbar1,fbar2,gbar2,pi)

  first= pi(:,:,fbar1,gbar1);
  second= pi(:,:,fbar2,gbar2);

  result= first*2+second;

  comparetradeareas=mesh(result);

end
```

fcdf.m

```
% cdf of buyers valuation, characterised by fbar

function fcdfc= fcdf (fbar,v)

fcdfc=fbar*v+v^2*(1-fbar);

end
```

fpdf.m

```
% pdf of buyers valuation, characterised by fbar

function fpdfc= fpdf(fbar,v)
fpdfc= fbar+v*2*(1-fbar);
end
```

gcdf.m

```
% cdf of sellers valuation, characterised by gbar
```

```
function gcdfc= gcdf (gbar,c)
gcdfc=gbar*c+c^2*(1-gbar);
end
```

gpdf.m

```
% pdf of seller's valuation, characterised by gbar

function gpdfc= gpdf(gbar,c)
gpdfc= gbar+c*2*(1-gbar);
end
```

main.m

```
clear all;

tic;            % chronometer mini-code for the elapsed time - toc is the
                % second part; at the end of the document.

beg= 0.001;     % beginning value for the discretised value range of seller
                % and buyer
fin= 0.999;     % ending value for the discretised value range of seller
                % and buyer
incr= 0.002;    % increment value for the discretised value range of the
                % seller and buyer

                % this value of the increment creates intervals each of
                % which is 0.002 unit lenght.

charbeg= 0.25;  % beginning value for the discretised characterising value
                % range of linear distributions
charfin= 1.75;  % ending value for the discretised characterising value
                % range of linear distributions
charincr= 0.25; % increment value for the discretised characterising value
                % range of linear distributions

                % characterising values represent the y-intercept of pdf.

                % I use alfa in order to refer lambda in the paper.
alfabeg=0.05;   % beginning value for the discretised value range of lambda
alfafin=2;      % ending value for the discretised value range of lambda
alfaincr=0.05;  % increment value for the discretised value range of lambda



v=beg:incr:fin; % generation of dicretised values of buyer
c=beg:incr:fin; % generation of dicretised values of seller

fbar=charbeg:charincr:charfin; % generation of dicretised characterising
                               % values of buyer's value distribution
gbar=charbeg:charincr:charfin; % generation of dicretised characterising
                               % values of seller's value distribution

alfa=alfabeg:alfaincr:alfafin; % generation of discretised lambdas

sumvecc= zeros (size(fbar,2), size (gbar,2), size (alfa,2));
% the vector I have created for integration (incentive-budget constraint)
```

```
sumvekk= zeros (size(fbar,2), size (gbar,2), size (alfa,2));
% the vector I have created for integration (incentive-budget constraint)
% - for equilibrium counterpart

 counter= size(fbar,2)*size(gbar,2)*size(alfa,2);
% I use counter in order to be able to monitor the duration of the progress

norm=(1/((size(v,2))^2)); % normalisation for integration

 for fbars=1:size(fbar,2)
     for gbars=1: size(gbar,2)
          for alfas= 1:1:size(alfa,2)
                               counter=counter-1
             for vs=1:size(v,2)
                for cs=1:size(c,2)

pi=pfunction(vs,cs,fbars,gbars,alfas,alfa,fbar,gbar,v,c);
% p(v,c) in the notes
pieq=pfunctioneq(vs,cs,fbars,gbars,alfas,alfa,fbar,gbar,v,c);
% p(v,c) in the notes

fdist=fpdf(fbar(fbars),v(vs)); % value of pdf of v.
gdist=gpdf(gbar(gbars),c(cs)); % value of pdf of c.
fcum=fcdf(fbar(fbars),v(vs));  % value of cdf of f.
gcum=gcdf(gbar(gbars),c(cs));  % value of cdf of c.

phis=(v(vs)/gdist)-((1-fcum)/(fdist*gdist));
%phi function in the paper
phiss=v(vs)-((1-fcum)/fdist);
%phi function in the paper-for eqm counterpart


gammas=(c(cs)/fdist)+((gcum)/(fdist*gdist));
%gamma function in the paper.
gammass=c(cs)+((gcum)/gdist);
%gamma function in the paper.-for eqm counterpart

sumvecc (fbars,gbars,alfas)=sumvecc(fbars,gbars,alfas)...
                            +(phis-gammas)*pi*fdist*gdist*norm ;
                            % integration

sumvekk (fbars,gbars,alfas)=sumvekk(fbars,gbars,alfas)...
                            +(phiss-gammass)*pieq*fdist*gdist*norm ;
                            % integration-for eqm counterpart

                end
             end
         end
     end
 end

 % the following loop is in order to see how many times the integration
 % (as a function of lambda) intersects with the horizontal axis.

 maximand=zeros(size(fbar,2), size (gbar,2));
```

```
for fbars=1:size(fbar,2)
    for gbars=1: size(gbar,2)
         for alfas= 2:1:size(alfa,2)

         if sumvecc(fbars,gbars,alfas-1)>0 && sumvecc(fbars,gbars,alfas)<0
            maximand(fbars,gbars)=maximand(fbars,gbars)+1;
         end

         if sumvecc(fbars,gbars,alfas-1)<0 && sumvecc(fbars,gbars,alfas)>0
            maximand(fbars,gbars)=maximand(fbars,gbars)+1;
         end

         end
    end
end

% now we find the lambda which makes the integration closest to zero
% for each fbar and gbar.

[minimisedvalues, indicesofbestalfas]=min(abs(sumvecc),[],3);

[minimisedvaluess, indicesofbestalfass]=min(abs(sumvekk),[],3);





% this following step generates the pi matrices, which will be employed
% for 2-dimensional graphs for trading regions for each fbar&gbar.

pi=zeros(size(v,2),size(c,2),size(fbar,2),size(gbar,2));

pis=zeros(size(v,2),size(c,2),size(fbar,2),size(gbar,2));
% for eqm counterpart

for fbars=1:size(fbar,2)
    for gbars=1: size(gbar,2)
         for vs=1:size(v,2)
               for cs=1:size(c,2)

 % for lk thinking
 bestalfa=indicesofbestalfas(fbars,gbars);

 pi(size(v,2)-vs+1,cs,fbars,gbars)=...
 pfunction(vs,cs,fbars,gbars,bestalfa,alfa,fbar,gbar,v,c);

 % and for eqm counterpart
 bestalfas=indicesofbestalfass(fbars,gbars);

 pis(size(v,2)-vs+1,cs,fbars,gbars)=...
 pfunctioneq(vs,cs,fbars,gbars,bestalfas,alfa,fbar,gbar,v,c);

               end
         end

    end
```

```
 end

% now we find ex-ante probability of trade and expected total surplus for
% each binary of fbar and gbar; for lk model

exanteprobtrade=zeros(size(fbar,2),size(gbar,2));
expectedtotalsurplus=zeros(size(fbar,2),size(gbar,2));

                          % and for eqm.

exanteprobtrades=zeros(size(fbar,2),size(gbar,2));
expectedtotalsurpluss=zeros(size(fbar,2),size(gbar,2));

for fbars=1:size(fbar,2)
     for gbars=1: size(gbar,2)
          for vs=1:size(v,2)
                for cs=1:size(c,2)

bestalfa=indicesofbestalfas(fbars,gbars);
bestalfas=indicesofbestalfass(fbars,gbars);

ppi=pfunction(vs,cs,fbars,gbars,bestalfa,alfa,fbar,gbar,v,c);
% p(v,c) in the paper.
ppis=pfunctioneq(vs,cs,fbars,gbars,bestalfas,alfa,fbar,gbar,v,c);
% p(v,c) in the paper.

fdist=fpdf(fbar(fbars),v(vs)); % value of pdf of v.
gdist=gpdf(gbar(gbars),c(cs)); % value of pdf of c.

exanteprobtrade(fbars,gbars)=...
   exanteprobtrade(fbars,gbars)+fdist*gdist*ppi*norm;
expectedtotalsurplus(fbars,gbars)=...
   expectedtotalsurplus(fbars,gbars)+ fdist*gdist*ppi*(v(vs)-c(cs))*norm;

exanteprobtrades(fbars,gbars)=...
   exanteprobtrades(fbars,gbars)+fdist*gdist*ppis*norm;
expectedtotalsurpluss(fbars,gbars)=...
   expectedtotalsurpluss(fbars,gbars)+ fdist*gdist*ppis*(v(vs)-c(cs))*norm;

                end
          end
     end
end

% this last piece of code is for saving trading regions,
% distribution functions and publishing the code

for fbars=1:size(fbar,2)
     for gbars=1: size(gbar,2)

         ef=100*fbar(fbars);
         gi=100*gbar(gbars);

         name1= num2str(ef);
         name2= num2str(gi);
         namel1 = strcat(name1,name2,'L1');
         nameeqm = strcat(name1,name2,'eqm');
```

```
        l1= blackandwhite(fbars,gbars,pi);
        eqm=blackandwhites(fbars,gbars,pis);


        saveas(l1,namel1,'png');
        saveas(eqm,nameeqm,'png');
        yeni (fbars,gbars,fbar,gbar)

    end
end

toc;
```

pfunction.m

```
% this is the value of a p(v,c), in lk model, for particular values of
% v,c,fbar,gbar,alfa

function pfuncc=pfunction (vs,cs,fbars,gbars,alfas,alfa,fbar,gbar,v,c)

fdist=fbar(fbars)+v(vs)*2*(1-fbar(fbars));         % pdf of v.
gdist=gbar(gbars)+c(cs)*2*(1-gbar(gbars));         % pdf of c.
fcum=fbar(fbars)*v(vs)+v(vs)^2*(1-fbar(fbars));    % cdf of f.
gcum=gbar(gbars)*c(cs)+c(cs)^2*(1-gbar(gbars));    % cdf of c.


               x=  v(vs)-c(cs) ;

               y= (v(vs)/gdist)-((1-fcum)/(fdist*gdist));

               z= (c(cs)/fdist)+(gcum/(fdist*gdist));


               r= x+(alfa(alfas))*(y-z);

                  if r>=0
                      pfuncc=1;
                  else
                      pfuncc=0;
                  end

end
```

pfunctioneq.m

```
% this is the value of a p(v,c), in usual model, for particular values of
% v,c,fbar,gbar,alfa

function pfuncc=pfunctioneq (vs,cs,fbars,gbars,alfas,alfa,fbar,gbar,v,c)

fdist=fbar(fbars)+v(vs)*2*(1-fbar(fbars));         % pdf of v.
gdist=gbar(gbars)+c(cs)*2*(1-gbar(gbars));         % pdf of c.
fcum=fbar(fbars)*v(vs)+v(vs)^2*(1-fbar(fbars));    % cdf of f.
gcum=gbar(gbars)*c(cs)+c(cs)^2*(1-gbar(gbars));    % cdf of c.
```

```
            x=  v(vs)-c(cs) ;

            y= (v(vs))-((1-fcum)/(fdist));

            z= (c(cs))+(gcum/(gdist));

            r= x+(alfa(alfas))*(y-z);

               if r>=0
                   pfuncc=1;
               else
                   pfuncc=0;
               end

end
```

sidebyside.m

```
%this is the function for visualisizng trade zone in lk model side by side

function [blackandwhite middle blackandwhites]=...
    sidebyside (fbars,gbars,pi, pis,fbar,gbar)

figure;
minuspi=1-pi(:,:,fbars,gbars);
minuspis=1-pis(:,:,fbars,gbars);

subplot(1,3,1);

blackandwhite=imagesc(minuspi);
title('trade zone in L1 model');
 set( gca,'XTick',0:50:450,'XTickLabel',{'0','0.1', '0.2','0.3', '0.4',...
     '0.5',  '0.6', '0.7', '0.8', '0.9',},'YTick',0:50:450,'YTickLabel',...
     {'1','0.9','0.8', '0.7','0.6', '0.5', '0.4',  '0.3', '0.2', '0.1'} );

axis image;

subplot(1,3,3);

blackandwhites=imagesc(minuspis);
 set( gca,'XTick',0:50:450,'XTickLabel',{'0','0.1', '0.2','0.3', '0.4',...
     '0.5',  '0.6', '0.7', '0.8', '0.9',},'YTick',0:50:450,'YTickLabel',...
     {'1','0.9','0.8', '0.7','0.6', '0.5', '0.4',  '0.3', '0.2', '0.1'} );

title('trade zone in equilibrium');

axis image;

subplot(1,3,2);
i = linspace(0,1);
buyer= fbar(fbars)+i*2*(1-fbar(fbars));
seller= gbar(gbars)+i*2*(1-gbar(gbars));

middle=plot(i,buyer,'-',i,seller,':');
title(' buyer (-) and seller (..) ');
axis image;
```

```
colormap gray;

end
```

yeni.m

```
%this is the function for visualising distributions

function [middle]= yeni (fbars,gbars,fbar,gbar)

h=figure;

         ef=100*fbar(fbars);
         gi=100*gbar(gbars);

         name1= num2str(ef);
         name2= num2str(gi);

         f=fbar(fbars);
         g=gbar(gbars);

         namedensity=strcat(name1,name2,'density');

i = linspace(0,1);
buyer= f+i*2*(1-f);
seller= g+i*2*(1-g);

middle=plot(i,buyer,'-',i,seller,':','linewidth',3);

title(' buyer (-) and seller (..) ');

ylim([0,2]);

colormap gray;

saveas(h,namedensity,'png');

end
```

blackandwhite.m

```
%this is the function for visualisizng trade zone in lk model

function blackandwhite= blackandwhite (fbar,gbar,pi)

figure;
minuspi=1-pi(:,:,fbar,gbar);

blackandwhite=imagesc(minuspi);

 set( gca,'XTick',0:50:450,'XTickLabel',{'0','0.1', '0.2','0.3', '0.4',...
     '0.5',  '0.6', '0.7', '0.8', '0.9',},'YTick',0:50:450,...
     'YTickLabel',{'1','0.9','0.8', '0.7','0.6', '0.5', '0.4',...
     '0.3', '0.2', '0.1'} );

colormap gray;

end
```